ENGINEERING MATHEMATICS-I

Prof. C. Nahak,

Prof. J. Kumar

Prof. S. Kumar,

ENGINEERING MATHEMATICS – I (3+0)

Course Developer

Prof. C. Nahak, Prof. J. Kumar & Prof. S. Kumar

Mathematics Department, IIT Kharagpur, Kharagpur 721 302)

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Lesson 1

Rolle's Theorem, Lagrange's Mean Value Theorem , Cauchy's Mean Value Theorem

1.1 Introduction

In this lesson first we will state the Rolle's theorems, mean value theorems and study some of its applications.

Theorem 1. 1 [Rolle's Theorem]: Let f be continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). If $f(a)=f(b)$, then there exists at least one number c in (a,b) such that $f'(c)=0$.

Proof: Assume $f(a)=f(b)=0$. If $f(a)=f(b)=k$ and $k=0$, then we consider $f(x)-k$ instead of $f(x)$. Since $f(x)$ is continuous on $[a,b]$ it attains its bounds: Let M and m be both maximum and minimum of $f(x)$ on $[a,b]$. If $M=m$, then $f(x)=m$ is throughout i.e., $f(x)$ is constant on $[a,b] \Rightarrow f'(x)=0$ for all x in $[a,b]$. Thus $\exists$ at least one c such that $f'(c)=0$.

Suppose $M \neq m$. If $f(x)$ varies on (a,b) then there are points where $f(c)>0$ or points where $f(c)<0$ .Without loss of generality assume $M>0$ and the

function takes the maximum value at $x = c$, so that $f(c) = M$. It is to be noted that if $c = a$, $f(c) = f(a) = 0 = f(b)$, which is a contradiction. Now as $f(c)$ is the maximum value of the function, it follows that $f(c + \Delta x) - f(c) \leq 0$, both when $\Delta x > 0$ and $\Delta x < 0$.

Hence,

$$\frac{f(c + \Delta x) - f(c)}{\Delta x} \leq 0$$

when $\Delta x > 0$

$$\frac{f(c+\Delta x)-f(c)}{\Delta x} \geq 0$$

when $\Delta x < 0$. Since it is given that the derivative at $x = c$ exists, we get $f'(c) \leq 0$ when $\Delta x > 0$ and $f'(c) \geq 0$ when $\Delta x < 0$. Combining the two inequalities we have, $f'(c) = 0$.

Note: Rolle's theorem shows that b/w any two zero's of a function $f(a)$ there exists at least one zero o $\acute{f}(x)$ i.e., $f(a) = f(b)$ clearly f is continous on [-1,1]

Example 1: Verify the Roll's theorem for $f(x) = x^2 \quad for\ all\ \ x \in [-1,1]$.

Solution:

(i) $f(1) = f(-1) = 1$, (ii) f is differentiable on $[-1,1]$, so all conditions of Roll's theorems are satisfying. Hence $f'(c) = 2c = 0$ implies $c = 0$ and $c \in (-1,1)$.

Example 2: $f(x) = 1 - |x|$ in $[-1,1]$.

Solution:

$f(-1) = f(1) = 0$, f is continuous. But $f(x)$ is not differentiable at $x = 0$. Note that $f'(x) \neq 0$, for which $f(x)$ is differentiable. As $f'(x) = -1$, for $x > 0$ and $f'(x) = 1$, for $x < 0$.

Example 3: Show that the equation $3x^5 + 15x - 8 = 0$, has only one real root

Solution:

$f(x) = 3x^5 + 15x - 8$ is an odd degree polynomial, hence it has at least one real root as complex roots occurs in pair.

Suppose $\exists$ two real roots x_1, x_2 such that $x_1 < x_2$, then on $[x_1, x_2]$, all properties of Roll's theorem satisfied, hence $\exists\, c \in (x_1, x_2)$, such that $f'(c) = 0$,

But $f'(x) = 15x^4 + 15 = 15(x^4 + 1) > 0, \quad for\ all\ x$, a contradiction to Rolle's therorem. Hence the equation has only one real root.

1.2. Mean Value Theorems

Theorem 1.2 [Lagrange's Mean Value Theorem]: If a function $f(x)$ is continuous on $[a,b]$, differentiable (a,b), then there exists at least one point c, $a < c < b$ such that $f(b) - f(a) = f'(c)(b-a)$. Hence Lagrange's mean value theorem can be written as

$f(b) - f(a) = hf'(a + \theta h)$, where $h = b - a\,; 0 \leq \theta \leq 1$.

Geometrical Representation: If all points of the arc AB there is a tangent line, then there is a point C between A and B at which the tangent is parallel to the chord connecting the points A and B.

1.2.1 Cauchy's Mean Value Theorem

Cauchy's mean value theorem, also known as the extended mean value theorem, is the more general form of the mean value theorem.

Theorem 1.2 [Cauchy's Mean Value Theorem]: It states that if functions f and g are both continuous on the closed interval $[a,b]$, and differentiable on the

open interval (a,b) and $g(a) \neq g(b)$ then there exists some $c \in (a,b)$, such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Note 1: Cauchy's mean value theorem can be used to prove L'Hospital's rule. The mean value theorem (Lagrange) is the special case of Cauchy's mean value theorem when $g(t) = t$.

Note 2: The proof of Cauchy's mean value theorem is based on the same idea as the proof of the mean value theorem

1.2.2 Another form of the statement: If $f(x)$ and $g(x)$ are derivable in $[a,a+h]$ and $g'(x) \neq 0$ for any $x \in [a,a+h]$, then there exists at least one number $\theta \in (0,1)$ such that

$$\frac{f(a+h)-f(a)}{g(a+h)-g(a)} = \frac{f'(a+\theta h)}{g'(a+\theta h)} \quad (0 < \theta < 1)$$

Example 4: Write the Cauchy formula for the functions $f(x) = x^2$, $g(x) = x^3$ on $[1,2]$.

Solution:

Clearly $f and\ g\ \ are\ continous\ and\ diff.on\ [1,2]\ \ g'(x) = 3x^2 = 0$ iff

$x = 0$, $0 \notin [1,2]$. $f'(x) = 2x$. Hence $g(1) \neq g(2)$

$$\frac{f(2)-f(1)}{g(2)-g(1)} = \frac{f'(c)}{g'(c)}.$$

i.e., $\frac{4-1}{8-1} = \frac{2c}{3c^2}$ implies $\frac{3}{7} = \frac{2}{3c}$, so $c = \frac{14}{9}$.

1.2.3 The Intermediate Value Theorem It states the following: If $y = f(x)$ is continuous on $[a,b]$, and N is a number between $f(a)$ and $f(b)$, then there is a $c \in [a,b]$ such that $f(c) = N$.

1.2.4 Applications of the Mean Value Theorem to Geometric properties of Functions.

Let f be a function which is continuous on a closed inteval $[a,b]$ and assume f. has a derivative at each point of the open interval (a,b). Then we have

1. (i) If $f'(x) > 0 \quad for\ all\ \ x \in (a,b)$, f is strictly increasing on $[a,b]$.
2. (ii) If $f'(x) < 0 \quad for\ all\ \ x \in (a,b)$, f is strictly decreasing on $[a,b]$.
3. (iii) If $f'(x) = 0 \quad for\ all\ \ x \in (a,b)$, f is constant.

Intermediate value Theorem for Derivatives: If $f'(x)$ exists for $a \leq x \leq b$, with $f'(a) \neq f'(b)$ then for any number d between $f'(a)$ and $f'(b)$ there is a number $a < c < b$ where $f'(c) = d$.

Application: If $f'(x)$ exists with $f'(x) \neq 0$, on any interval then f has a differentiable inverse, there.

Converse of Rolle's theorem : - (need not true).

Example 1.5 Let $f(x)$ be continuous on $[a,b]$ and differentiable (a,b). If $\exists c \in (a,b)$ such that $f'(c) = 0$, does it follow that $f(a) = f(b)$?

Solution:

No: Take for example $f(x) = x^2$ on $[-1,2]$, $f'(x) = 2x = 0$ implies $x = 0$.

But $f(-1) = 1$ and $f(2) = 4$.

Example 1.6 Show that $|\sin x - \sin y| \leq |x - y|$

Solution:

Let $f(t) = \sin t$ on $[y, x]$, By mean value theorem $\sin x - \sin y = f'(c)(x - y)$,

But $f'(t) = \cos t$, and $|\cos t| \leq 1$, for all t. Hence

$|\sin x - \sin y| = |f'(c)(x - y)| \leq |x - y|$.

Example 1.7 Show that $\tan^{-1} x_2 - \tan^{-1} x_1 < x_2 - x_1$, for all $x_2 > x_1$.

Solution:

Let $f(x) = \tan^{-1} x$ on $[x_1, x_2]$. By mean value theorem $\tan^{-1} x_2 - \tan^{-1} x_1 =$

$f'(c)(x_2 - x_1) = \frac{1}{1+c^2}(x_2 - x_1)$ but $\frac{1}{1+c^2} < 1$ for all c. Hence the results.

Questions: Answer the following question.

1. Verify the truth of Rolle's theorem for the functions

   (a) $f(x) = x^2 - 3x + 2$ on $[1,2]$

   (b) $f(x) = (x-1)(x-2)(x-3)$ on $[1,3]$

   (c) $f(x) = \sin x$ on (a) $[0, \pi]$

2. The function $f(x) = 4x^3 + x^2 - 4x - 1$ has roots 1 and -1. Find the root of the derivative $f'(x)$ mentioned in Rolle' s throrem.

3. Verify Lagrange's formula for the function $f(x) = 2x - x^2$ on $[0,1]$.

4. Apply Lagrange theorem and prove the inequalities

   (i) $e^x \geq 1 + x$ (ii) $\ln(1+x) < x \ (x > 0)$

(iii) $b^n - a^n < nb^{n-1}(b-a)$ for $(b > a)$

5. Using Cauchy's mean value theorem show that $\lim_{x\to 0} \frac{\sin x}{x} = 1$

Keywords: Rolle's Theorem, Lagrange's and Cauchy's mean value; L'Hospital's rule; Intermediate value.

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Suggested Readings

Tom, M. Apostol. (2003). Calculus, Volume II Second Editions, Publishers, John Willey & Sons, Singapore.

Lesson 2

Taylor's theorem / Taylor's expansion, Maclaurin's expansion

2.1 Introduction

In calculus, Taylor's theorem gives us a polynomial which approximates the function in terms of the derivatives of the function. Since the derivatives are usually easy to compute, there is no difficulty in computing these polynomials.

A simple example of Taylor's theorem is the approximation of the exponential function e^x near $x = 0$.

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdot \frac{x^n}{n!}$$

The precise statement of the Taylor's theorem is as follows:

Theorem 2.1: If $n \geq 0$ is an integer and f is a function which is n times continuously differentiable on the closed interval $[a, x]$ and $n+1$ times differentiable on the open interval (a, x), then

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a)$$

$$+\frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

Here, n! denotes the factorial of n, and $R_n(x)$ is a remainder term, denoting the difference between the Taylor polynomial of degree n and the original function. The remainder term $R_n(x)$ depends on x and is small if x is close enough to a.

Several expressions are available for it. The Lagrange form is given by

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} = \mathrm{a} + \theta(x-a)$$

where $0 < \theta < 1$.

If we put $a = 0$, Taylor's formula reduces to Maclaurin's formula.

where ξ lies between a and x.

Notes

- In fact, the mean value theorem is used to prove Taylor's theorem with the Lagrange remainder term.
- The Taylor series of a real function $f(x)$ that is infinitely differentiable in a neighborhood of a real number a, is the power series of the form

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

- In general, a function need not be equal to its Taylor series, since it is possible that the Taylor series does not converge, or that it converges to a different function.
- However, for some functions $f(x)$, one can show that the remainder term $R_n(x)$ approaches zero as n approaches ∞. Those functions can be expressed as a Taylor series in a neighbourhood of the point a and are called analytic.

Example 2.1 Show that $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

Solution:

Here $f(x) = \sinh x$, $f'(x) = \cosh x$, So

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots$$

$$f(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$R_n(x) = \frac{h^n}{n!}f^{(n)}(a + \theta h)$. But for $a = 0$ and $h = x$

$$|R_n(x)| = |\frac{x^n}{n!}f^{(n)}(\theta x)|$$

$$\lim_{n\to\infty}|R_n| = \lim_{n\to\infty}\left|\frac{x^n}{n!}\right||cosh(\theta x)| = 0$$

Example 2.2 . Find the Taylor series expansion of $\frac{1}{x^2-4}$

Solution: $f(x) = \frac{1}{x^2-4} = \frac{1}{(x+2)(x-2)}$

$$= \frac{A}{x+2} + \frac{B}{x-2}$$

$$= -\frac{1}{4(x+2)} + \frac{1}{4(x-2)}$$

$$= -\frac{1}{8(1+\frac{x}{2})} + \frac{1}{-8(1-\frac{x}{2})}$$

$$= -\frac{1}{8}(1+\frac{x}{2})^{-1} - \frac{1}{8}(1-\frac{x}{2})^{-1}$$

for $|\frac{x}{2}| < 1$, we have

$$= -\frac{1}{8}[1 - \frac{x}{2} + (-\frac{x}{2})^2 + (-\frac{x}{2})^3 \cdots]$$

$$-\frac{1}{8}[1 + \frac{x}{2} + (\frac{x}{2})^2 + (\frac{x}{2})^3 \cdots]$$

$$= -\frac{1}{8}[2 + (\frac{x}{2})^2 + \cdots]$$

Example 2.3 : Find $f^{(100)}(0)$ if $f(x) = e^{x^2}$

Ans: $f^{(100)}(0) = \frac{100!}{50!}$.

Questions: Answer the following questions.

1. Expand in power of $x-2$ of the polynomial $x^4 - 5x^3 + 5x^2 + x + 2.$
2. Expand in power of $x+1$ of the polynomial $x^5 + 2x^4 - x^2 + x + 1.$
3. Write Taylor's formula for the function $y = \sqrt{x}$ when $a = 1, n = 3.$
4. Write the Maclaurin formula for the function $y = \sqrt{1+x}$ when $n = 2.$
5. Using the results of above problem, estimate the error of the approximate equation $\sqrt{1+x} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2$ when $x = 0.2.$
6. Write down the Taylor's expansion for the function $f(x) = \sin x$ about the point $a = \frac{\pi}{4}$ with $n = 4.$
7. Applying Taylor's theorem with remainder prove that $1 + \frac{x}{2} - \frac{x^2}{8} < \sqrt{1+x} < 1 + \frac{x}{2}$ if $x > 0.$
8. Applying Maclaurin's theorem with remainder expand

   (i) $\ln(1+x)$ (ii) $(1+x)^m.$

Keywords: Taylor's Formula, Taylor's Series, Maclaurin Formula and Series.

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Suggested Readings

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Lesson 3

Indeterminate forms ; L'Hospital's Rule

3.1 Introduction

Consider the following limits $\lim_{x\to 4}\frac{x^2-16}{x-4}$ and $\lim_{x\to\infty}\frac{4x^2-5x}{1-3x^2}$

In the first limit if we put $x=4$ we will get $\frac{0}{0}$ and in the second limit if we "plugged" in infinity we get $\frac{\infty}{-\infty}$ (recall that as x goes to infinity a polynomial will behave in the same fashion that it's largest power behaves). Both of these are called Indeterminate form.

3.1.1 Indeterminate forms

First limit can be found by the factorizing the numerator cancelling the common factor. That is

$$\lim_{x\to 4}\frac{x^2-16}{x-4}$$

$$=\lim_{x\to 4}\frac{(x-4)(x+4)}{x-4}$$

$$=\lim_{x\to 4}(x+4)$$

$$=8$$

The second limit can be evaluated as:

$$\lim_{x\to\infty}\frac{4x^2-5x}{1-3x^2}$$

$$=\lim_{x\to\infty}\frac{4-\frac{5}{x}}{\frac{1}{x^2}-3}$$

$$=-\frac{4}{3}$$

However what about the following two limits. $\lim_{x\to 0}\frac{\sin x}{x}$ and $\lim_{x\to\infty}\frac{e^x}{x^2}$, This first is a $\frac{0}{0}$ indeterminate form, but we can't factor this one. The second is an $\frac{\infty}{\infty}$ indeterminate form, but we can't just factor an x^2 out of the numerator. Does there exists some method to evaluate the limits? The answer is yes. By (L'Hospital's Rule).

Suppose that we have one of the following cases,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}\text{ or }\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\pm\infty}{\pm\infty}$$

where a can be any real number, infinity or negative infinity. In these cases we have,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Theorem 3.1: Suppose the functions $f(x)$ and $g(x)$ in $[a,b]$, satisfy the Cauchy Theorem and $f(a) = g(a) = 0$, then if the ratio $\frac{f'(x)}{g'(x)}$ has a limit as $x \to a$, there also exists $\lim_{x\to a} \frac{f(x)}{g(x)}$, and $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = A$.

Proof.: On the interval $[a,b]$ take some point $x \neq a$. Applying the Cauchy's mean value theorem we have

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(\xi)}{g'(\xi)}$$

where ξ is a number lies between a and x. But it is given that $f(a) = g(a) = 0$ and so

$$\frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)} \quad \text{.....(1)}$$

If $x \to a$, then $\xi \to a$, since ξ lies between x and a. Suppose if $\lim_{x\to a} \frac{f'(x)}{g'(x)} = A$, by (1) $\lim_{\xi\to a} \frac{f'(\xi)}{g'(\xi)}$ exists and is equal to A. Hence

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(\xi)}{g'(\xi)}$$

$$=\lim_{\xi\to a}\frac{f'(\xi)}{g'(\xi)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=A$$

and, finally,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Note 3.1: The theorem also holds for the case where the functions $f(x)$ and $g(x)$ are not defined at $x=a$, but $\lim_{x\to a}f(x)=\lim_{x\to a}g(x)=0$. We can make them to be continuous at $x=a$ by redefine $f(a)=\lim_{x\to a}f(x)=0$, $g(a)=\lim_{x\to a}g(x)=0$, since $\lim_{x\to a}\frac{f(x)}{g(x)}$ does not depend on whether the function $f(x)$ and $g(x)$ are defined at $x=a$.

Note 3.2: If $f'(a)=g'(a)=0$ and the derivatives $f'(x)$ and $g'(x)$ satisfy the conditions that we imposed by the theorem on the functions $f(x)$ and $g(x)$, then applying the L'Hospital rule $\lim_{x\to a}\frac{f'(x)}{g'(x)}=\lim_{x\to a}\frac{f''(x)}{g''(x)}$, and so forth.

Note 3.3: If $g'(x) = 0$, but $f'(x) \neq 0$, then the theorem is applicable to the reciprocal ratio $\frac{g(x)}{f(x)}$, which tends to zero as $x \to a$. Hence, the ratio $\frac{f(x)}{g(x)}$ tends to infinity.

Example 3.1: $\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x} = \lim_{x\to 0} \frac{e^x + e^{-x} - 2}{1 - \cos x}$

$$= \lim_{x\to 0} \frac{e^x - e^{-x}}{\sin x} = \lim_{x\to 0} \frac{e^x + e^{-x}}{\cos x}$$

$$= \frac{2}{1} = 2.$$

Note 3.4: The L'Hospital rule is also applicable if $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to\infty} g(x) = 0$.

Put $x = \frac{1}{z}$, we see that $z \to 0$ as $x \to \infty$ and therefore $\lim_{z\to 0} f(\frac{1}{z}) = 0$, and $\lim_{z\to 0} g(\frac{1}{z}) = 0$. Applying the L'Hospital rule to the ratio

$\frac{f(\frac{1}{z})}{g(\frac{1}{z})}$, we find that

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{z\to 0}\frac{f(\frac{1}{z})}{g(\frac{1}{z})}$$

$$=\lim_{z\to 0}\frac{f'(\frac{1}{z})(-\frac{1}{z^2})}{g'(\frac{1}{z})(-\frac{1}{z^2})}$$

$$=\lim_{z\to 0}\frac{f'(\frac{1}{z})}{g'(\frac{1}{z})}=\lim_{x\to\infty}\frac{f'(x)}{g'(x)}$$

which proves the results.

We also stated in earlier that if both $f(x)$ and $g(x)$ approaching infinity as $x\to a$ (or $x\to\infty$), the L'Hospital rule is also applied.

Example 3.2: Find $\lim_{x\to\frac{\pi}{2}}\frac{\tan x}{\tan 3x}(\frac{\infty}{\infty})$

Solution:

Taking derivative both numerator and denominator five times we obtain: **Ans: 3**

Other Indeterminate forms :

The other indeterminate forms reduce to the following cases. (a) $0.\infty$ (b) 0^0 (c) ∞^0 (d) 1^∞ (e)$\infty-\infty$.

(a) Let $\lim_{x\to a} f(x) = 0, \lim_{x\to a} g(x) = \infty$, it is required to find $\lim_{x\to a}[f(x)g(x)]$,

i.e. the indeterminate form $0.\infty$. Now

$$\lim_{x\to a}[f(x)g(x)]$$

$$= \lim_{x\to a} \frac{f(x)}{\frac{1}{g(x)}}$$

or $f(x)g(x) = \frac{g(x)}{\frac{1}{f(x)}}$ If $\lim f(x) = \infty, x \to a$ & $\lim g(x) = 0, x \to a$

which is $(\frac{0}{0})$- form or one can write

$$\lim_{x\to a} \frac{g(x)}{\frac{1}{f(x)}}$$

$(\frac{\infty}{\infty})$- form

Example 3.3

$$\lim_{x\to 0} x^n \ln x = \lim_{x\to 0} \frac{\ln x}{\frac{1}{x^n}}$$

$$= \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{n}{x^{n+1}}} = \lim_{x\to 0} \frac{x^n}{n} = 0$$

b) Let $\lim_{x\to a} f(x) = 0$, $\lim_{x\to a} g(x) = 0$, it is required to find $\lim_{x\to a}[f(x)]^{g(x)}$. Put $y = [f(x)]^{g(x)}$. Taking logarithms of both sides of it, we have

$$\ln y = g(x)[\ln f(x)]$$

$$\lim_{x\to a} \ln y = \ln \lim_{x\to a} y$$

(by the continuity of $\ln y$) and if $\ln \lim_{x\to a} y = b \Rightarrow e^b = \lim_{x\to a} y$.

Similarly we can find the Indeterminate form $\infty^0, 1^\infty$

Example 3.4: $\lim_{x\to 0} x^x$ Solution: Put $y = x^x$,

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} x\ln x$$

$$= \lim_{x\to 0} (x\ln x) = \lim_{x\to 0} \frac{\ln x}{\frac{1}{x}}$$

$$= 0$$

So $\lim_{x\to 0} y = e^0 = 1$.

Example 3.5: Find the $\lim_{x\to 0} \left(\frac{1}{x}\right)^{\tan x}$

Ans: 1

Example 3.6 Using Taylor's formula compute

$$\lim_{x\to 0}\frac{x-\sin x}{e^x-1-x-\frac{x^2}{2}}$$

Ans: 1

Questions: Answer the following questions.

Evaluate the following limits :

1. $\lim_{x\to 1}\frac{x-1}{x^n-1}$

2. $\lim_{x\to 0}\frac{e^{x^2}-1}{\cos x-1}$

3. $\lim_{x\to 0}\frac{\sin x}{\sqrt{1-\cos x}}$

4. $\lim_{x\to 0}\frac{e^y+\sin y-1}{\ln(1+y)}$

5. $\lim_{x\to 1}\frac{\ln(x-1)-x}{\tan\frac{\pi}{2x}}$

6. $\lim_{x\to 1}\left[\frac{x}{x-1}-\frac{1}{\ln x}\right]$

7. $\lim_{x\to 0}(\cot x)^{\frac{1}{\ln x}}$

8. $\lim_{x\to 0}\left(\frac{1}{x}\right)^{tanx}$

Ans.: 1. 1, 2. -2, 3. Limit does not exist, 4. 2, 5. 0, 6. $\frac{1}{2}$, 7. $\frac{1}{6}$ & 8. 1

Keywords: Indeterminate forms ; L'Hospital's Rule.

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Lesson 4

Limit, Continuity of Functions of Two Variables

4.1 Introduction

So far we have studied functions of a single (independent) variables. Many familiar quantities, however, are functions of two or more variables. For instance, the work done by the force $(W = F.D)$ and the volume of the rigid circular cylinder ($V = \pi r^2 h$) are both functions of two variables. The volume of a rectangular solid ($V = xyz$) a function of three variables. The notation for a function of two or more variables is similar to that for a function of single variable.

Example 4.1: $z = f(x, y) = x^2 + xy$ (two variables)

Example 4.2: $w = f(x, y, z) = x + 2y - 3z$ (three variable)

A function f of two variables is a rule that assigns a real number $z = f(x, y)$ to each ordered pair (x, y) of real numbers in the domain of f. The range of f is the set of all values of the function: $\{z | z = f(x, y)$ where $(x, y) \in D\}$.

In concrete terms: A function $z = f(x, y)$ is usually just a formula involving the two variables x and y. For every x and y we put in, we get a number z out. The set of all (x, y) we allowed to put into the function is called the domain of the function. Usually the domain is unspecified, and then the domain is the set of all (x, y) we can put into the formula for f and not get square roots of negatives, or division by zero, or some such. i.e.,the domain is usually the set of all (x, y) we can put into the function without getting an undefined expression.
This is the natural domain. The range is simply all the numbers z we can "hit" by putting all (x, y) from the domain into the function.

Example: 4.3: Let $f(x, y) = \sqrt{49 - x^2 - y^2}$. The domain is the disk of radius 7, centre at origin. Now $49 - x^2 - y^2$ will be bigger if x, y ar each smaller. So $f(x, y)$ is biggest when $x = y = 0$. This is $f(0,0) = 7$. Now the smallest value can achieve is 0, when $49 - x^2 - y^2 = 0$ (which happens, for example when $x = 7$ and $y = 0$). If $49 - x^2 - y^2 < 0$, f could not be defined. Hence the range is [0,7].

Definition. The graph of a function f of two variables is the set

$$\{(x,y,z)|z = f(x,y)\ for\ some\ \ (x,y) \in D\},$$

where D is the domain of f. That is, the graph is the surface $z = f(x,y)$ in 3-dimensinal Euclidean Space $\mathbb{R}^3$.

4.1.1 A contour curves or level curves

A contour curve for a function $z = f(x,y)$ is a trace of the surface $z = f(x,y)$ parallel to the xy-plane. That is, let $z = k$ for some number k, and plot $k = f(x,y)$ in the xy-plane.

The domain of a function of two variables f, which is denoted $dom(f)$ from now onwards is the set of all points (x,y) in the xy-plane for which $f(x,y)$ is defined. For example, $A = \{(x,y)|x > y\}$ means that A is the set of points (x,y) such that x is greater than y.

Example 4.4. Determine the domain of $f(x,y) = \ln(y - 2x)$

Solution:

Since the argument of $ln(.)$ must be positive, the domain of f is the set of points (x,y) for which the denominator is not equal to 0. However, $y-2x>0$ means that $y>2x$. In set notation this is written as $dom(f)=\{(x,y)|y>2x\}$.

Most of the sets in the xy-plane we encounter will be bounded by a closed curve.

As a result, we define an open region to be the set of all points inside of but not including a closed curve, and we define a closed region to be the set of all points inside of and including a closed curve.

Equivalently, a point (p,q) is said to be a boundary point of a set S if any circle centered at (p,q) contains both points inside of and outside of S, and correspondingly, a set S is open if it contains none of its boundary points and closed if it contains all of its boundary points.

Example 4.5. Determine if the domain of the following function is open or closed. $f(x,y)=\sqrt{9-x^2-y^2}$

Solution:

To begin with, the quantity $9 - x^2 - y^2$cannot be negative since it is under the square root. Thus, the domain of f is the set of points that satisfy

$$9 - x^2 - y^2 \geq 0 \text{ or } 9 \geq x^2 + y^2.$$

That is, the domain is the set of points (x, y) inside and on the circle of radius 3 centered at the origin, which we write as $dom(f) = \{(x, y) | x^2 + y^2 \leq 9\}$.

Moreover, the domain is a closed region of the xy-plane since it contains the boundary circle of radius 3 centered at the origin.

We say that a region S is connected if any two points in S can be joined by a curve which is contained in S:

4.1.2 Functions of Space and Time

Functions of two variables are important for reasons other than that their graph is a surface. In particular, a function of the form $u(x,t)$ is often interpreted to be a function of x at a given point in time. For example, let's place an xy-coordinate system on a violin whose strings have a length of l, If $u(x,t)$ is considered the displacement of a string above or below a horizontal line at a point x and at a time t, then $y = u(x,t)$ is the shape of the string at a fixed time t.

Likewise, *u* (*x*, *t*) might represent the temperature at a distance *x*from one end of the rod at time t.

4.1.3 Limits and Continuity

Now we will extend the properties of limits and continuity from the familiar function of one variable to the new territory of functions of two or more variables.

Let us recall limit of function of single variable: Let f be a function defined on an open interval containing a (except possible at a) and let L be a real number.

The statement $\lim_{x\to a} f(x) = L$ means that for given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$, whenever $|x - a| < \delta$.

In less formal language this means that, if the limit holds, then $f(x)$ gets closer and closer to L as x gets closer and closer to a.

Consider the following limits.

$$\lim_{x\to -2} \frac{x-2}{x^2-4} = \frac{-2-2}{(-2)^2-4} = \frac{-4}{0} \to ?$$

Good job if you saw this as “limit does not exist” indicating a vertical asymptote at $x = -2$.

$$\lim_{x\to 2}\frac{x-2}{x^2-4}=\frac{2-2}{(2)^2-4}=\frac{0}{0}\to ?$$

This limit is indeterminate. With some algebraic manipulation, the zero factors could cancel and reveal a real number as a limit. In this case, factoring leads to……

$$\lim_{x\to 2}\frac{x-2}{x^2-4}=\lim_{x\to 2}\frac{x-2}{(x+2)(x-2)}$$

$$=\lim_{x\to 2}\frac{1}{(x+2)}=\frac{1}{4}$$

The limit exists as x approaches 2 even though the function does not exist. In the first case, zero in the denominator led to a vertical asymptote; in the second case the zeros cancelled out and the limit reveals a hole in the graph at $(2,\frac{1}{4})$.

The concept of limits in two dimensions can now be extended to functions of two variables.

Definition 4.1 Let f be a function of two variables defined on an open disc centered at (x_0,y_0) i.e., $\{(x,y)|\sqrt{(x-x_0)^2+(y-y_0)^2}<r^2\}$, except polssible

at (x_0,y_0), and let L be the real numbers Then

$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ if given $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$|f(x,y) - L| < \epsilon$ whenever $\sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$.

Graphically for any point $(x,y) \neq (x_0,y_0)$ in the disc with radiusδ, the value $f(x,y)$ lies between $L - \epsilon$ and $L + \epsilon$.

Example 4.6 Let $z = f(x,y) = x^2 + y^2 + 3$.

For the limit of this function to exist at (-1,3), values of z must get closer to 13 as points (x,y) on the xy-plane get closer and closer to (-1,3). $\lim_{(x,y)\to(-1,3)} f(x,y) = 13$. For proof we have to go back to epsilon and delta.

Example 4.7 Verifying the limit by definition $\lim_{(x,y)\to(a,b)} x = a$.

Solution:

We have to show that $|x-a| < \varepsilon$ whenever $\sqrt{(x-a)^2 + (y-b)^2} < \delta$. Now

$|x-a| \leq \sqrt{(x-a)^2} \leq \sqrt{(x-a)^2 + (y-a)^2} < \delta$. Let $\delta = \varepsilon$.

Example 4.8. Show that $\lim_{(x,y)\to(0,0)} \frac{5x^2y}{x^2+y^2} = 0$.

Solution:

Now

$$\left|\frac{5x^2y}{x^2+y^2}\right| = 5|y|\left(\frac{x^2}{x^2+y^2}\right) \leq 5|y|$$

$$\leq 5\sqrt{x^2+y^2} < 5\delta.$$

Put $\delta = \frac{\varepsilon}{5}$, whenever $\sqrt{(x-0)^2+(y-0)^2} < \delta$.

Example 4.9.

Solution:

To show that $|(2x-3y)-(-4)| < \varepsilon$, whenever

$$\sqrt{(x-1)^2+(y-2)^2} < \delta.$$

Now $|2x-3y+4| = |2(x-1)-3(y-2)| \leq 2|x-1|+3|y-2|$

$< 2\delta + 3\delta = 5\delta$. Set $\delta = \frac{\varepsilon}{5}$.

For a single variable function we have $\lim_{x\to a} f(x)$ has two direction i.e.,

$$\lim_{x\to a^+} f(x) \text{ and } \lim_{x\to a^-} f(x).$$

But in case of function of two variables the $\lim_{(x,y)\to(a,b)} f(x,y)$, (x,y) approaches to (a,b) in infinitely many directions.

Example 4.10: Test whether $\lim_{(x,y)\to(0,0)}(\frac{x^2-y^2}{x^2+y^2})^2$ exists.

Solution:

Let $(x,y)\to(0,0)$ on the line $y=mx$. So

$$\lim_{(x,y)\to(0,0)}(\frac{x^2-y^2}{x^2+y^2})^2 = \lim_{x\to 0}(\frac{x^2-m^2x^2}{x^2+m^2x^2})^2$$

$$= (\frac{1-m^2}{1+m^2})^2.$$

As depend on m, so the limit does not exist.

Example 11: Solution: Let $x=r\cos\theta$, $y=r\sin\theta$, $(x,y)\to(0,0)$ implies $r\to 0$. The limit becomes $\lim_{r\to 0}\frac{\sin r^2}{r^2} = \lim_{t\to 0}\frac{\sin t}{t} = 1$.

Definition of Continuity of a Function of Two Variables

A function of two variables is continuous at a point (a,b) in an open region S if $f(a,b)$ is equal to the limit of $f(x,y)$ as (x,y) approaches (a,b). In limit notation:

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

Give Definition

The function f is continuous in the open region S if f is continuous at every point in S.

The following results are presented without proof. As was the case in functions of one variable, continuity is "user friendly". In other words, if k is a real number and f and g are continuous functions at (a,b) then the functions below are also continuous at (a,b):

$$kf(x,y) = k[f(x,y)], (f \pm g)(x,y) = f(x,y) \pm g(x,y),$$

$$(fg)(x,y) = f(x,y)g(x,y), \left(\frac{f}{g}\right)(x,y)$$

$$= \frac{f(x,y)}{g(x,y)} \text{ if } g(a,b) \neq 0.$$

The conclusions indicate that arithmetic combinations of continuous functions are also continuous —that polynomial and rational functions are continuous on their domains.

Finally, the following result asserts that the composition of continuous functions are also continuous. If f is continuous at (a,b) and g is continuous at $f(a,b)$,

then the composition function $(gof)(x,y) = g(f(x,y))$ is continuous at (a,b)

and

$$\lim_{(x,y)\to(a,b)} g(f(x,y)) = g(f(a,b)).$$

Example 4.12 Find the limit and discuss the continuity of the function $\lim_{(x,y)\to(1,2)} \frac{x}{\sqrt{2x+y}}$

Solution:

$\lim_{(x,y)\to(1,2)} \frac{x}{\sqrt{2x+y}} = \frac{1}{\sqrt{2(1)+2}} = \frac{1}{2}$. The function will be continuous when

$2x + y > 0$.

Example 4.13. Using ε and δ show that the function $f(x,y) = x^3 - 3xy^2$ is

continuous at origin.

Solution:

Set $x = r\cos\theta$ and $y = r\sin\theta$ (θ is fixed). Then

$|f(x,y)| = r^3|\cos^3\theta - 3\cos\theta\sin^2\theta| < 4r^3$. Take $r = \sqrt{x^2+y^2} < \delta = (\frac{\varepsilon}{4})^{\frac{1}{3}}$.

Example 4.14. Is it possible to define $f(x,y) = \frac{x^3+y^3}{x^2+y^2}$ at $(0,0)$ so that $f(x,y)$ is continuous?

Solution:

Note that

$$|\frac{x^3+y^3}{x^2+y^2}| \leq \frac{|x|^3}{x^2+y^2} + \frac{|y|^3}{x^2+y^2} = \frac{x^2|x|}{x^2+y^2} + \frac{y^2|y|}{x^2+y^2}$$

$$\leq |x| + |y| \leq 2\sqrt{x^2+y^2} = \varepsilon$$

where $\sqrt{x^2+y^2} < \delta$ and $\delta = \frac{\varepsilon}{2}$. If we define $f(0,0) = 0$, $f(x,y)$ is continuous every where.

Example 4.15. Show that the function $\frac{\sin xy}{\sqrt{x^2+y^2}}$ is continuous if we define $f(0,0) = 0$.

Solution:

Discontinuity possible only at $(0,0)$. Note with $x = r\cos\theta$ and $y = r\sin\theta$, from $|\sin\alpha| < |\alpha|$ for small α , that $|\frac{\sin xy}{\sqrt{x^2+y^2}}| < r$; hence limit at $(0,0)$ exists and is 0.

Property 1: If a function $f(x,y)$ is defined and continuous in a closed and bounded domain D, then there will be at least one point (x^*,y^*) in D such that

$$f(x^*,y^*) \geq f(x,y).$$

And at least one point $f(x_*,y_*) \in D$ such that

$$f(x_*,y_*) \leq f(x,y).$$

We call $f(x^*, y^*) = M$ as the maximum value of the function and $f(x_*, y_*) = m$ is the minimum value of the function. This result states that a function which is continuous on a closed and bounded domain D has a maximum and minimum.

Property 2: If $f(x, y)$ has both maximum and minimum M and m respectively, let $m < \mu < M$, then $\exists\ (x_1, y_1) \in D$ such that $f(x_1, y_1) = \mu$.

Corollary to property 2.
If a function $f(x, y)$ is continuous in a closed and bounded domain D and assumes both positive and negative values, then there will be a point inside the domain at which the $f(x, y)$ vanishes.

Questions: Answer the following questions.

1. Find $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^3+y^3}$, if it exists.

2. Show that $\lim_{(x,y)\to(0,0)} \frac{x+y}{x^2+y^2+1} = 0$

3. Prove that $\lim_{(x,y)\to(1,2)} 2x - 3y = -4$.

4. Find $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$

5. Find $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^3+y^3}$, if it exists.

6. Test for continuity $(a)\, f(x,y) = \frac{x-2y}{x^2+y^2}$ $(b)\, g(x,y) = \frac{2}{y-x^2}$

7. Find the $\lim_{(x,y)\to(0,1)} \frac{\sin^{-1}\left(\frac{x}{y}\right)}{1+xy}$ and discuss the continuity of the function $\frac{\sin^{-1}\left(\frac{x}{y}\right)}{1+xy}$ at $0,1)$.

8. Find the $\lim_{(x,y)\to(0,0)} \frac{-1}{2}\ln(x^2+y^2)$,

and discuss the continuity of the function $\frac{-1}{2}\ln(x^2+y^2)$ at $(0,0)$.

Example 1: Let $f(x,y) = \frac{xy}{x^2+y^2}$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$ for $(x,y) = (0,0)$. Is it continuous at $(0,0)$ or can we make continuous by redefining $f(0,0)$? (Hint: not possible)

Example 2: Is it possible to extend $f(x,y) = \frac{x+y}{x^2+y^2}$ to the origin so that the resulting function is continuous? (Hint: not possible)

Keywords: Limit, Continuity, Maximum and Minimum values.

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Lesson 5

Partial and Total Derivatives

5.1 Introduction

Let $z = f(x,y)$, we denote $\frac{\partial z}{\partial x}$ as the partial derivative of z with respect to x and

define as

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

and similarly $\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x,y+\Delta y) - f(x,y)}{\Delta y}$

Example 5.1: Given $z = x^y$, find the partial derivative of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

Solution:

$\frac{\partial z}{\partial x} = yx^{y-1}$, $\frac{\partial z}{\partial y} = x^y \ln x$.

The partial derivatives of a function of any number of variables are determined similarly. Thus if $u = f(x,y,z,t)$

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y, z, t) - f(x,y,z,t)}{\Delta x}$$

$$\frac{\partial u}{\partial t} = \lim_{\Delta t \to 0} \frac{f(x,y,z,t+\Delta t) - f(x,y,z,t)}{\Delta t}$$

Informally, we say that the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point (x_0, y_0, z_0) denote the slope of the surface in the x- and y-directions, respectively.

Example 5.2: Find the slopes of the surface given by $f(x,y) = -\frac{x^2}{2} - y^2 + \frac{25}{8}$ at the point $(\frac{1}{2}, 1, 2)$ in the x-direction and the y-direction.

Solution:

$$\frac{\partial f}{\partial x}|_{(\frac{1}{2},1,2)} = -x|_{(\frac{1}{2},1,2)} = -\frac{1}{2}$$

$$\frac{\partial f}{\partial y}|_{(\frac{1}{2},1,2)} = -2y|_{(\frac{1}{2},1,2)} = -2.$$

5.1.2 Differentiability for Functions of Two Variables

We begin by reviewing the concept of differentiation for functions of one variable. We define the derivative in case of function of single variable.

Let $f: D \subset R \mapsto R$ and let a be an interior point of D. Then f is differentiable at a means

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=f'(a)$$

or equivalently

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=f'(a)$$

exists. The number $f'(a)$ is called the derivative of f at a.

Geometrically the derivative of a function at a is interpreted as the slope of the tangent line to the graph of f at the point $(a, f(a))$.

Extending the definition of differentiability in its present form to functions of two variables is not possible because the definition involves division and dividing by a vector or by a point in two dimensional space is not possible. To carry out the extension, an equivalent definition is developed that involves division by a distance. The limit statement can be rewritten as

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}-f'(a)=0 \quad or$$

$$\lim_{x\to a}\frac{f(x)-f(a)-(x-a)f'(a)}{|x-a|}=0$$

So the following definition is equivalent to the original one.

Let $f: D \subset \mathrm{R} \mapsto R$ and let a be an interior point of D. Then f is differentiable at a means there is a number, $f'(a)$, such that

$$\lim_{x \to a} \frac{f(x) - f(a) - (x-a)f'(a)}{|x-a|} = 0.$$

One way to interpret this expression is that $f(x) - f(a) - (x-a)f'(a)$ tends to 0 faster than $|x-a|$ and consequently $f(x)$ is approximately equal to $f(a) + (x-a)f'(a)$. The equation $y = f(a) + (x-a)f'(a)$ is the equation of the line tangent to the graph of f at the point $(a, f(a))$. So $f(x)$ is approximated very well by its tangent line. This observation is the bases for linear approximation.

Using this form of the definition as a model it is possible to construct a definition of differentiability for functions of two variables.

Definition 5.1. Let $f: D \subset R^2 \mapsto R$ and let (x_0, y_0) be an interior point of D.

Then f is differentiable at (x_0, y_0) means there are two numbers, $f_x(x_0, y_0) = f_x()$ and $f_y(x_0, y_0) = f_y()$ such that

$$\lim_{(x,y)\to(x_0,y_0)} = \frac{f(x,y) - f(x_0,y_0) - (x-x_0)f_x() - (y-y_0)f_y()}{\sqrt{(x-x_0)^2 + (y-y_0)^2}}$$

$$= 0.$$

The vector

$$f_x(x_0, y_0)\vec{i} + f_y(x_0, y_0)\vec{j}, \ \vec{i} = (1,0), \ \vec{j} = (0,1)$$

or

$$(f_x(x_0, y_0), f_y(x_0, y_0))$$

is called the derivative of f at the point (x_0, y_0). Interpret this definition as requiring that the graph of f has a tangent plane at the point $(x_0, y_0, f(x_0, y_0))$.

In fact it is easy to get an equation for this tangent plane. It is $z = f(x_0, y_0) + (x - x_0)f_x(x_0, y_0) + (y - y_0)f_y(x_0, y_0)$. In

$f_x(x, y) = \frac{\partial}{\partial x} f(x, y)$, the same symbol x is use for two different purposes. First

as a subscript where it denotes the variable of differentiation and second as the first coordinate of a point in $\mathbb{R}^2$. Strictly speaking such a dual use of one symbol

is improper, but this is so common as to be acceptable. In the general case, the derivative is a vector in n space and it is computed by computing all of the first

order partial derivatives. As in the case of functions of one variable, differentiability implies continuity.

For functions of one variable if the derivative, $f(x)$, can be computed, then f is

differentiable at x. The corresponding assertion for functions of two variables is

false, as we know existence of partial derivative does not mean the function of two variable is continuous. We might suspect that if f is continuous at (x_0, y_0)

and the first order partial derivatives exist there, then f is differentiable at (x_0, y_0) but that conjecture is false as the following example shows.

Example 5.1. Let $f(x,y) = \frac{xy}{\sqrt{x^2+y^2}}$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$.

Solution:

So if f were differentiable at (0, 0),we would have that

$lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\sqrt{x^2+y^2}} = 0$ as $f_x(0,0) = 0$ and $f_y(0,0) = 0$. That is

$lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = 0$. But if the limit is computed along the path $y = x$, we get $lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$.

The natural question to ask then is under what conditions can we conclude that f is differentiable at (x, y). The answer is contained in the following theorem.

Theorem 5.1. Let $f: D \subset \mathbb{R}^2 \mapsto \mathbb{R}$ and let P_0 be an interior point of D. Suppose all of the first order partial derivatives of f exist in a open disk about $P_0 = (x_0, y_0)$ and are continuous at P_0. Then f is differentiable at P_0.

Example 5.2. Show that the function $f(x, y) = \ln(x^2 + y^2)$ is differentiable everywhere in its domain.

Solution:

The domain of f is all of $\mathbb{R}^2$ except for the origin. We shall show that f has continuous partial derivatives everywhere in its domain (that is, the function f is in C^1). The partial derivatives are $f_x = \frac{2x}{x^2+y^2}$ and $f_y = \frac{2y}{x^2+y^2}$. Since each of f_x and f_y is the quotient of continuous functions, the partial derivatives are

continuous everywhere except the origin (where the denominators are zero). Thus, f is differentiable everywhere in its domain.

We know that if a function is differentiable at a point, it has partial derivatives there. Therefore, if any of the partial derivatives fail to exist, then the function cannot be differentiable. This is what happens in the following example.

Example 5.3: Consider the function $f(x,y) = \sqrt{x^2 + y^2}$. Is it differentiable at the origin.

Solution:

Let us find the partial derivatives if they exist at $(0,0)$. Now

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x,0) - f(0,0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{\Delta^2 x + 0 - 0}}{\Delta x} = \lim_{\Delta x \to 0} \frac{|\Delta x|}{\Delta x}.$$

Since the limit does not exit so $f_x(0,0)$ does not exit. Similarly we can show also $f_y(0,0)$ does not exist. Thus f cannot be differentiable at the origin.

In Example 5.3 the partial derivatives f_x and f_y did not exist at the origin and this was sufficient to establish non differentiability there.

In the following example even if both of the partial derivatives, $f_x(0,0)$ and $f_y(0,0)$, exist f is not differentiable at (0, 0).

Example 5.4: Consider the function $f(x,y) = x^{\frac{1}{3}}y^{\frac{1}{3}}$. Show that the partial derivatives $f_x(0,0)$ and $f_y(0,0)$ exist, but that f is not differentiable at (0, 0).

Solution:

Now $$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x,0)-f(0,0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{0-0}{\Delta x} = 0,$$

and similarly $f_y(0,0) = 0$. Suppose the function is differentiable at (0,0),

i.e., $$lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\sqrt{x^2+y^2}} = 0$$

That is $$lim_{(x,y)\to(0,0)} \frac{x^{\frac{1}{3}}y^{\frac{1}{3}}}{\sqrt{x^2+y^2}} = 0.$$

If this limit exists, we get the same value no matter how x and y approach 0.

Suppose we take $y = x > 0$. Then the limit becomes

$$lim_{(x,y)\to(0,0)} \frac{x^{\frac{1}{3}} y^{\frac{1}{3}}}{\sqrt{x^2+y^2}}$$

$$= \lim_{x\to 0} \frac{x^{\frac{2}{3}}}{x\sqrt{2}} = \lim_{x\to 0} \frac{1}{x^{\frac{1}{3}}\sqrt{2}}.$$

But this limit does not exist, since small values for x will make the fraction arbitrarily large. Thus, this function is not differentiable at the origin, even though the partial derivatives $f_x(0,0)$ and $f_y(0,0)$ exist.

In summary if a function is differentiable at point, then it is continuous there. Having both partial derivatives at a point does not guarantee that a function is continuous there.

Theorem 5.1 : $f(x,y), g(r,s), h(r,s) \in C^1$

$$\Rightarrow \frac{\partial}{\partial r} f(g,h) = f_1(g,h)g_1(r,s) + f_2(g,h)h_1(r,s)$$

$$\frac{\partial}{\partial s} f(g,h) = f_1(g,h)g_2(r,s) + f_2(g,h)h_2(r,s).$$

Here subscript 1 and 2 denote the partial derivative with respect to its first and second argument, respectively. The proof is given in Lesson 7.

5.1.2 Total Differential

Definition 5.2 (Total Differential) For a function of two variables, $z = f(x, y)$

if Δx and Δy are given increments and, then the corresponding increment of z is

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y).$$

The differentials dx and dy are independent variables; that is, they can be given any values. Then the differential dz, also called the total differential, is defined by

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

Example 5.5: If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz.

Further, if x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz. Which is easier to compute Δz or dz?

Solution:

By definition,

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = (2x + 3y)dx + (3x - 2y)dy.$$

Putting $x = 2$, $dx = \Delta x = 0.05$, $y = 3$, and $dy = \Delta y = -0.04$, we get

$$dz = [2(2) + 3(3)]0.05 + [3(2) - 2(3)](-0.04) = 0.65$$

The increment of z is $\Delta z = f(2.05,2.96) - f(2,3)$

$$[(2.05)^2 + 3(2.05)(2.96) - (2.96)^2] - [2^2 + 3(6) - 3^2]$$

$$= 0.6449$$

Notice that $\Delta z \approx dz$ but dz is easier to compute.

5.2 Total derivative: In the mathematical field of differential calculus, the term total derivative has a number of closely related meanings.

The total derivative of a function, f, of several variables, e.g., t, x, y, etc., with respect to one of its input variables, e.g., t, is different from the partial derivative. Calculation of the total derivative of f with respect to t does not assume that the other arguments are constant while t varies; instead, it allows the other arguments to depend on t. The total derivative adds in these indirect

dependencies to find the overall dependency of f on t. For example, the total derivative of $f(t,x,y)$ with respect to t is

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Consider multiplying both sides of the equation by the differential dt .

The result will be the differential change df in the function f. Because f depends on t, some of that change will be due to the partial derivative of f with respect to t. However, some of that change will also be due to the partial derivatives of f with respect to the variables x and y. So, the differential is applied to the total derivatives of x and y to find differentials dx and dy, which can then be used to find the contribution to df.

Example 5. 6: Find the total derivative of $z = x^2 + \sqrt{y}$, $y = \sin x$

Solution:

$\frac{\partial z}{\partial x} = 2x$, $\frac{\partial z}{\partial y} = \frac{1}{2\sqrt{y}}$, $\frac{dy}{dx} = \cos x$.

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}$$

$$= 2x + \frac{1}{2\sqrt{y}}\cos x$$

$$= 2x + \frac{1}{2\sqrt{\sin x}}\cos x.$$

Questions: Answer the following questions.

1. Test the differentiability of $f(x,y) = \sqrt{y^2 - x^2}$

2. Find the total differential of $z = \tan^{-1}\left(\frac{x}{y}\right), (x,y) \neq (0,0)$

3. $u = xz + \frac{x}{z}$, $z \neq 0$

4. Find $\frac{df}{dt}$ at $t = 0$ where $f(x,y) = x\cos y + e^x \sin y, x = t^2 + 1, y = t^3 + t$

Keywords: Partial Derivative, Differential, Total Differential

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Suggested Readings

Tom M. Apostol. (2003). Calculus, Volume II Second Editions, Publishers,John Willey & Sons, Singapore.

Lesson 6

Homogeneous Functions, Euler's Theorem

6.1 Introduction

A polynomial in x and y is said to be homogeneous if all its terms are of same degree. For example,

$$f(x,y) = x^2 - 2xy + 3y^2$$

is homogeneous. It is easy to generalize the property so that functions not polynomials can have this property.

Definition 6.1

A function $f(x,y)$ is homogeneous of degree n in a region D iff, for $(x,y) \in D$ and for every positive value λ, $f(\lambda x, \lambda y) = \lambda^n f(x,y)$. The number n is +ve, -ve, or zero and need not be an integer.

Example 6.1 $f(x,y) = x^{\frac{1}{3}} y^{-\frac{4}{3}} \tan^{-1}\left(\frac{y}{x}\right)$. Here $n = -1$; D is any quadrant without the axes.

Example 6.2 $f(x,y) = 3 + \ln\left(\frac{y}{x}\right)$

This function is homogeneous of degree 0; D is first and third quadrant without the axes.

Example 6.3 $f(x,y) = x^{\frac{1}{3}}y^{-\frac{2}{3}} + x^{\frac{2}{3}}y^{-\frac{1}{3}}$.

This function is not homogeneous.

Theorem 6.1 [Euler's Theorem] Let $f(x,y)$ is a homogeneous function of degree n in R (region) and f_x and f_y are continuous in R. Then

$$f_x(x,y)x + f_y(x,y)y = nf(x,y)$$

for all $(x,y) \in R$.

Proof. Now differentiate $f(\lambda x, \lambda y) = \lambda^n f(x,y)$ partially with respect to λ, we obtain

Chain rule : $$xf_1(\lambda x, \lambda y) + yf_2(\lambda x, \lambda y) = n\lambda^{n-1}f(x,y).$$

Finally set $\lambda = 1$.

Example 6.4 If $u = \sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$. Then show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{\cos 2u \sin u}{4\cos^3 u}$$

Proof. Let $w = \sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}} = f(x,y)$.

u is not homogeneous function , but w is

$$f(\lambda x, \lambda y) = \frac{\lambda x + \lambda y}{\sqrt{\lambda}(\sqrt{x}+\sqrt{y})} = \frac{\lambda(x+y)}{\sqrt{\lambda}(\sqrt{x}+\sqrt{y})} = \lambda^{\frac{1}{2}} \frac{x+y}{\sqrt{x}+\sqrt{y}} = \lambda^{\frac{1}{2}} f(x,y)$$

w is homogeneous function of degree $\frac{1}{2}$.Therefore

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = \frac{1}{2}w = \frac{1}{2}\sin u$$

But $\frac{\partial w}{\partial x} = \cos u \frac{\partial u}{\partial x}, \frac{\partial w}{\partial y} = \cos u \frac{\partial u}{\partial y}$

Hence $x\cos u \frac{\partial u}{\partial x} + y\cos u \frac{\partial u}{\partial y} = \frac{1}{2}\sin u$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u \quad \text{.....(6.1)}$$

Differentiating (1) partially w.r.t. x , we have

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\,\partial y} = \frac{1}{2}\sec^2 u \frac{\partial u}{\partial x}$$

$$\Rightarrow x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y} = (\frac{1}{2}\sec^2 u - 1)\frac{\partial u}{\partial x} \quad \text{.....(6.2)}$$

Differentiating (1) partially w.r.t. y, we have

$$y\frac{\partial^2 u}{\partial y^2}+x\frac{\partial^2 u}{\partial x\,\partial y}+\frac{\partial u}{\partial y}=\frac{1}{2}\sec^2 u\frac{\partial u}{\partial y}$$

$$\Rightarrow x\frac{\partial^2 u}{\partial x\,\partial y}+y\frac{\partial^2 u}{\partial y^2}=(\frac{1}{2}\sec^2 u-1)\frac{\partial u}{\partial y} \qquad \text{.....(6.3)}$$

Multiplying (2) by x, (3) by y and adding, we have

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}$$

$$=(\frac{1}{2}\sec^2 u-1)(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y})$$

$$=(\frac{1}{2\cos^2 u}-1)(\frac{1}{2}\tan u)$$

$$=-\frac{\cos 2u\sin u}{4\cos^3 u}$$

Example 6.5 If $u=\sin^{-1}\frac{x+2y+3z}{x^8+y^8+z^8}$. Then find

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}$$

Ans.: $(-7\tan u)$.

Example 6.6 (1) If $u = tan^{-1}\frac{x^3+y^3}{x-y}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$.

Solution:

Here u is not a homogenous function but $\tan u = \frac{x^3+y^3}{x-y}$ is a homogenous fucntion of degree 2

i.e., $x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = 2\tan u$

or $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\tan u \cdot \cos^2 u = \sin 2u$

(2) If $u = ln\frac{x^4+y^4}{x+y}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$.

Solution:

u is not homogenous function, but e^u is a homogenous function of degree 3 in x, y.

By Euler's theorem, we have $x\frac{\partial}{\partial x}(e^u) + y\frac{\partial}{\partial y}(e^u) = 3e^u$

$$xe^u\frac{\partial u}{\partial x} + ye^u\frac{\partial u}{\partial y} = 3e^u$$

i.e., $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3$

Questions: Answer the following questions.

1. If $u=sin^{-1}\left(\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right)$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}+3\tan u=0$

2. If $=f\left(\frac{y}{x}\right)$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$

3. If $=xf\left(\frac{y}{x}\right)$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$

4. If $u=sin^{-1}\frac{x}{y}+tan^{-1}\frac{y}{x}$, then find the value of $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$

Keywords: Homogeneous Function, Euler's Theorem, Parial Derivatives

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Lesson 7

Composite and Implicit Functions for Two Variables

7.1 Introduction

The chain rule works for functions of more than one variable. Consider the function $z = f(x,y)$ where $x = g(t)$ and $y = h(t)$, and $g(t)$ and $h(t)$ are differentiable with respect to t, then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Suppose that each argument of $z = F(u,v)$ is a two-variable function such that $u = h(x,y)$ and $v = g(x,y)$, and that these functions are all differentiable. Then the chain rule would look like:

$$\frac{\partial z}{\partial x} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y}.$$

If we consider $\vec{r} = (u,v)$ above as a vector function, we can use vector notation to write the above equivalently as the dot product of the gradient of F and a derivative of $\vec{r}$:

$$\frac{\partial F}{\partial x} = \nabla F . \partial \vec{r}$$

Partial and Total Increment: We consider a function $z = f(x,y)$, increase the independent variable x by Δx (keeping y fixed), then z will be increased: this

increase is called the partial increment with respect to x which we denote as $\Delta_x z$, so that

$$\Delta_x z = f(x + \Delta x, y) - f(x, y).$$

Similarly we define $\Delta_y z$. If we increase the argument x by Δx and y by Δy, we get z a new increment Δz, which is called the total increment of z and defined by

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y).$$

It is noted that total increment is not equal to the sum of the partial increments, $\Delta z \neq \Delta_x z + \Delta_y z$. Let us assume that $f(x, y)$ has continuous partial derivatives at the point (x, y) under consideration. Express Δz in terms of partial derivatives. To do this we have

$$\Delta z = [f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)] + [f(x, y + \Delta y) - f(x, y)]$$

and using Lagrange mean value theorem separately

$$\Delta z = \Delta x \frac{\partial f(\bar{x}, y + \Delta y)}{\partial x} + \Delta y \frac{\partial f(x, \bar{y})}{\partial y}.$$

(where $\bar{y}$ lies between y and $y + \Delta y$ and $\bar{x}$ between x and $x + \Delta x$). As partial derivatives are continuous it follows that

$$\Delta z = \frac{\partial f(x,y)}{\partial x}\Delta x + \frac{\partial f(x,y)}{\partial y}\Delta y + \psi_1 \Delta x + \psi_2 \Delta y.$$

Where the quantities $\psi_1(x,y)$ and $\psi_2(x,y)$ approach zero as Δx and Δy approach zero.

Now we will derive the total differential of composite function.

Theorem 7.1: $f(x,y), g(r,s), h(r,s) \in C^1$

$$\Rightarrow \frac{\partial}{\partial r} f(g,h) = f_1(g,h)g_1(r,s) + f_2(g,h)h_1(r,s)$$

$$\frac{\partial}{\partial s} f(g,h) = f_1(g,h)g_2(r,s) + f_2(g,h)h_2(r,s).$$

We use this formula for the composite function $f(x,y), x = \phi(r,s), y = \psi(r,s)$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

Example 7.1: $f(x,y) = xy, f_1 = y, f_2 = x$

$$\frac{\partial}{\partial r} gh = \frac{\partial}{\partial r} f(g(r,s), h(r,s))$$

$$= yg_1 + xh_1$$

$$= hg_1 + gh_1$$

We can generalize this results. If $w = F(z,u,v,s)$ is a function of four arguments z,u,v,s and each of them depends on x and y, then

$$\frac{\partial w}{\partial x} = \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial F}{\partial s}\frac{\partial s}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial F}{\partial s}\frac{\partial s}{\partial y}.$$

If a function $z = F(x,y,u,v)$, where y, u, v depend on a single independent variables x: $y = f(x)$, $u = \phi(x)$, $v = \psi(x)$, then z is actually a function of one variable x only.

Hence,

$$\frac{dz}{dx} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x}$$

$$+\frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x}$$

$$= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx}$$

$$+\frac{\partial F}{\partial u}\frac{du}{dx} + \frac{\partial F}{\partial v}\frac{dv}{dx}$$

This formula is known as the formula for calculating the total derivative $\frac{dz}{dx}$ (in contrast to the partial derivative $\frac{\partial z}{\partial x}$).

Example 7.2: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ of $z = \ln(u^2 + v)$, $u = e^{x+y^2}$, $v = x^2 + y$.

Solution:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}.$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}.$$

$$\frac{\partial u}{\partial x} = e^{x+y^2}, \frac{\partial u}{\partial y} = 2ye^{x+y^2}, \frac{\partial v}{\partial x} = 2x, \frac{\partial v}{\partial y} = 1, \frac{\partial z}{\partial u} = \frac{2u}{u^2+v}, \frac{\partial z}{\partial v} = \frac{1}{u^2+v}.$$

So

$$\frac{\partial z}{\partial x} = \frac{2u}{u^2+v}e^{x+y^2} + \frac{1}{u^2+v}2x$$

$$= \frac{2}{u^2+v}(ue^{x+y^2} + x)$$

$$\frac{\partial z}{\partial y} = \frac{2u}{u^2+v}2ye^{x+y^2} + \frac{1}{u^2+v}$$

$$= \frac{1}{u^2+v}(4uye^{x+y^2} + 1).$$

In these expressions, we have to substitute e^{x+y^2} and x^2+y for u and v. respectively.

Example 7.3: Find the total derivative of $z = x^2 + \sqrt{y}, y = \sin x$

Solution:

$\frac{\partial z}{\partial x} = 2x, \frac{\partial z}{\partial y} = \frac{1}{2\sqrt{y}}, \frac{dy}{dx} = \cos x.$

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}$$

$$= 2x + \frac{1}{2\sqrt{y}}\cos x$$

$$= 2x + \frac{1}{2\sqrt{\sin x}}\cos x.$$

7.1.1 Let us find the the total differential of the composite function $z = F(u, v)$ and $u = \phi(x, y)$ and $v = \psi(x, y)$, we know the total differential

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

Now substitute the expression $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial x}$ defined in the above composite function, after simplification we obtain

$$dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv.$$

Where $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$ and $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$

Example 7.4: Find the total differential of the composite function $z = u^2v^3$, $u = x^2\sin y$, $v = x^3e^y$.

Solution:

$$du = 2x\sin y dx + x^2\cos y dy$$

$$dv = 3x^2e^y dx + x^3e^y dy$$

$$dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv$$

$$= 2uv^3du + 3u^2v^2dv$$

$$= 2uv^3(2x\sin ydx + x^2\cos ydy)$$

$$+3u^2v^2(3x^2e^ydx + x^3e^y)dy$$

$$= (2uv^3.2x\sin y + 3u^2v^2.3x^2e^y)dx$$

$$+(2uv^3x^2\cos y + 3u^2v^2x^3e^y)dy$$

7.2 Composite and implicitly Functions:

Let some function y of x be defined by the equation $F(x,y) = 0$. We shall prove the following theorem.

Theorem 7.2 Let a function y of x be defined implicitly by the equation

$$F(x,y) = 0 \qquad (7.1)$$

where $F(x,y)$, $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$ are continuous in the domain D containing the point (x,y), which satisfies (7.1), also $\frac{\partial F}{\partial y} \neq 0$ at the point (x,y). Then

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}.$$

Proof. Given $F(x,y)$ is a function of two variables x, and y and y is again a function of x so that F is a composite function of x. Its derivative with respect to x is

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx}$$

$$= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx}.$$

As F is considered as a function of x alone, which is identically zero. So we have

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\frac{dy}{dx}=0,$$

which implies $\frac{dy}{dx}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$.

Example 7.5: An equation is given that connects x and y

$$e^y-e^x+xy=0,$$

find $\frac{dy}{dx}$.

Solution:

$F(x,y)=e^y-e^x+xy$, $\frac{\partial F}{\partial x}=-e^x+y$, $\frac{\partial F}{\partial y}=e^y+x$, by the above theorem we obtain $\frac{dy}{dx}=-\frac{-e^x+y}{e^y+x}$.

Questions: Answer the following questions.

1. Find $\frac{df}{dt}$ at $t=0$ where $f(x,y)=x^3+y^3$, $x=e^t$, $y=\cos t$.
2. If $z=f(x,y)$, $x=e^{2u}+e^{-2v}$, $y=e^{-2u}+e^{2v}$, then show that
3. $\frac{\partial f}{\partial u}+\frac{\partial f}{\partial v}=2[x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}]$.
4. Find $\frac{dy}{dx}$, when $f(x,y)=\ln(x^2+y^2)+tan^{-1}\left(\frac{y}{x}\right)=0$.

5. Find $\frac{dy}{dx}$, when $x^y + y^x = a$, a any constant, $x, y > 0$.

Keywords: Chain Rule, Composite Function

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Lesson 8

Derivative of Higher Order

8.1 Introduction

Derivative of higher order of composite function may be computed by the principles given in Lesson 7. As an example, let us compute three drivatives of order two for the function $u = f\big(\varphi(r,s),\psi(r,s)\big)$. We assume that three functions along with partial derivatives are continous upto order 3. First let us consider the higher order partial derivatives.

8.1.1 For $u = f(\emptyset(r,s),\psi(r,s)$, we assume that the three fucntions

$f,\emptyset,\psi \in C^2$.

$\frac{\partial u}{\partial r} = f_1\emptyset_1 + f_2\psi_1$, $\frac{\partial u}{\partial s} = f_1\emptyset_2 + f_2\psi_2$

Differentiating again, remember that f_1 *and* f_2 are themselves composite functions.

$$\frac{\partial^2 u}{\partial r^2} = f_1\emptyset_{11} + f_2\psi_{11+}\emptyset_1[f_{11}\emptyset_1 + f_{12}\psi_1] + \psi_1[f_{21}\emptyset_1 + f_{22}\psi_1]$$

$$\frac{\partial^2 u}{\partial s\partial r} = f_1\emptyset_{12} + f_2\psi_{12+}\emptyset_1[f_{11}\emptyset_2 + f_{12}\psi_2] + \psi_1[f_{21}\emptyset_2 + f_{22}\psi_2]$$

$$\frac{\partial^2 u}{\partial s^2} = f_1 \emptyset_{22} + f_2 \psi_{22+} \emptyset_2 [f_{11} \emptyset_2 + f_{12} \psi_1] + \psi_2 [f_{21} \emptyset_1 + f_{22} \psi_2]$$

We omit the arguments in these fucntions to have space. If we admit that

$f_{12} = f_{21}, \emptyset_{12} = \emptyset_{21}, \psi_{12} = \psi_{21}$ then it is easily shown that $\frac{\partial^2 u}{\partial r \partial s} = \frac{\partial^2 u}{\partial s \partial r}$.

8.1.1 Higher-order partial derivatives As is true for ordinary derivatives, it is possible to take second, third, and higher order partial derivatives of a function of several variables, provided such derivatives exist.

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}, \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}.$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{yx}, \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{xy}.$$

It is not true in general $f_{yx} = f_{xy}$

Example 8.1 Let $f(x,y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}$; for $(x,y) \neq (0,0)$ and $f(0,0) = 0$.

Solution:

We have

$$f_{xy}(0,0) = \lim_{\Delta x \to 0} \frac{f_y(0+\Delta x,0) - f_y(0,0)}{\Delta x}$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0,0+\Delta y) - f(0,0)}{\Delta y} = 0$$

$$f_y(\Delta x, 0) = \lim_{\Delta y \to 0} \frac{f(\Delta x,0+\Delta y) - f(\Delta x,0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{\Delta x \Delta y(\Delta x^2 - \Delta y^2)}{\Delta y(\Delta x^2 + \Delta y^2)} = \Delta x.$$

Hence

$$f_{xy}(0,0) = \lim_{\Delta x \to 0} \frac{\Delta x - 0}{\Delta x} = 1.$$

$$f_{yx}(0,0) = \lim_{\Delta y \to 0} \frac{f_x(0,0+\Delta y) - f_x(0,0)}{\Delta y}$$

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(0+\Delta x,0) - f(0,0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{0}{\Delta x} = 0$$

$$f_x(0,\Delta y) = \lim_{\Delta x \to 0} \frac{f(0+\Delta x,\Delta y) - f(0,\Delta y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x \Delta y(\Delta x^2 - \Delta y^2)}{\Delta x(\Delta x^2 + \Delta y^2)} = -\Delta y$$

So

$$f_{yx}(0,0) = \lim_{\Delta y \to 0} \frac{-\Delta y - 0}{\Delta y} = -1.$$

i.e., $f_{yx}(0,0) \neq f_{xy}(0,0)$.

8.1.2 Partial Derivatives of Higher Order (Equality of f_{xy} and f_{yx}).

If $f(x,y)$ possesses continuous second order partial derivatives f_{xy} and f_{yx}, then

$$f_{xy} = f_{yx}$$

Note: Existence of partial derivatives does not ensure continuity of a function.

Example 8.2 Let $f(x,y) = \frac{xy(x^2-y^2)}{x^2+y^2}$; for $(x,y) \neq (0,0)$ and $f(0,0) = 0$.

Solution:

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x,0) - f(0,0)}{\Delta x} = 0$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0,\Delta y) - f(0,0)}{\Delta y} = 0$$

But $f(x,y)$ is discontinuous at $(0,0)$.

Example 8.3 If $f(x,y) = g(x)h(y)$, show that $f_{xy} = f_{yx}$

Solution:

$$f_x(x,y) = g'(x)h(y), f_{yx} = g'(x)h'(y)$$

$$f_y(x,y) = g(x)h'(y), f_{xy} = g'(x)h'(y)$$

i.e., $f_{xy} = f_{yx}$.

Example 8.4 If $z = g(x)h(y)$, show that $z\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$

Solution: We have $\frac{\partial z}{\partial x} = g'(x)h(y)$ and $\frac{\partial z}{\partial y} = g(x)h'(y)$. Now

$\frac{\partial^2 z}{\partial x\,\partial y} = g'(x)h'(y)$.

So

$$\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = g(x)h(y)g'(x)h'(y)$$

$$= z\frac{\partial^2 z}{\partial x\,\partial y}$$

Example 8.5 Let $f(x,t) = u(x+at) + v(x-at)$, where u and v are assumed to have continuous second partial derivatives, show that $a^2 f_{xx} = f_{tt}$.

Solution:

$$f_x = u'(x+at) + v'(x-at), f_{xx} = u''(x+at) + v''(x-at)$$

$$f_t = au'(x+at) - av'(x-at), f_{tt} = a^2 u''(x+at)$$

$$+a^2 v''(x-at) = a^2 f_{xx}.$$

Questions: Answer the following questions.

1. For $u = f(g(t), h(t))$, find $\frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial t^2}$

2. Find $f''(t)$, if $f = e^x \sin y, x = t^2, y = 1 - t^2$ by not eleminating x and y.

3. Show that the functions $z = \phi(x^2 - y^2)$, where $\phi(u)$ is a differentiable function, satisfies the relationship $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0$.

4. Find the derivatives $\frac{dy}{dx}$ of the functions represented implicitly

$(i)\ \sin(xy) - e^{xy} - x^2 y = 0\ \ (ii)\ xe^y + ye^x - e^{xy} = 0\ \ (iii)\ y^x = x^y\ \ (iv)\ x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

5. If $r = x\phi(x+y) + y\psi(x+y)$, show that

$$\frac{\partial^2 r}{\partial x^2} - 2\frac{\partial^2 r}{\partial x \partial y} + \frac{\partial^2 r}{\partial y^2} = 0.$$

(ϕ and ψ are twice differentiable function.)

6. If $u = \frac{1}{y}[\phi(ax+y) + \phi(ax-y)]$, show that

$$\frac{\partial^2 u}{\partial x^2} = \frac{a^2}{y^2} \cdot \frac{\partial}{\partial y}\left(y^2 \frac{\partial u}{\partial y}\right).$$

Keywords: Higher order derivatives, higher order partial derivatives

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Suggested Readings

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Lesson 9

Taylor's expansion for function of two variables

9.1 Introduction

Let $z = f(x, y)$ which is continuous, together with all its partial derivatives up to $(n+1)$-th order inclusive, in some neighborhood of a point (a, b). Then like a function of single variable we can represent $f(x, y)$ as sum of an n-th degree polynomial in power of $(x-a)$ and $(y-b)$ and some remainder. We consider here in case $n = 2$ and show that $f(x, y)$ has of the form

$$f(x,y) = A_0 + D(x-a) + E(y-b)$$

$$+\frac{1}{2!}[A(x-a)^2 + 2B(x-a)(y-b) + C(y-b)^2] + R_2 \quad (1)$$

where A_0, D, E, A, B, C are independent of x and y, and R_2 is the remainder, and it is very similar to function of single variable.

Let us apply the Taylor formula for function $f(x, y)$ of the variable y assuming x to be constant.

$$f(x,y) = f(x,b) + \frac{y-b}{1} f_y(x,b)$$

$$+\frac{(y-b)^2}{1.2}f_{yy}(x,b)+\frac{(y-b)^3}{1.2.3}f_{yyy}(x,\eta_1) \quad (2)$$

where $\eta_1 = b+\theta_1(y-b)$, $0<\theta_1<1$. We expand the functions $f(x,b)$, $f_y(x,b)$, $f_{yy}(x,b)$ in a Taylor's series in powers of $(x-a)$

$$f(x,b) = f(a,b)+\frac{x-a}{1}f_x(a,b)$$

$$+\frac{(x-a)^2}{1.2}f_{xx}(a,b)+\frac{(x-a)^3}{1.2.3}f_{xxx}(\xi_1,b) \quad (3)$$

where $\xi_1 = x+\theta_2(x-a)$, $0<\theta_2<1$

$$f_y(x,b) = f_y(a,b)+\frac{x-a}{1}f_{yx}(a,b)$$

$$+\frac{(x-a)^2}{1.2}f_{yxx}(\xi_2,b) \quad (4)$$

where $\xi_2 = x+\theta_3(x-a)$, $0<\theta_3<1$

$$f_{yy}(x,b) = f_{yy}(a,b)+\frac{x-a}{1}f_{yyx}(\xi_3,b) \quad (5)$$

where $\xi_3 = x+\theta_4(x-a)$, $0<\theta_4<1$. Substituting expression (3), (4) and (5) into formula (2), we get

$$f(x,y) = f(a,b) + \frac{x-a}{1} f_x(a,b) + \frac{(x-a)^2}{1.2} f_{xx}(a,b)$$

$$+ \frac{(x-a)^3}{1.2.3} f_{xxx}(\xi_1, b) + \frac{y-b}{1} [f_y(a,b) + \frac{x-a}{1} f_{yx}(a,b)$$

$$+ \frac{(x-a)^2}{1.2} f_{yxx}(\xi_2, b)] + \frac{(y-b)^2}{1.2} [f_{yy}(a,b)$$

$$+ \frac{x-a}{1} f_{yyx}(\xi_3, b)] + \frac{(y-b)^3}{1.2.3} f_{yyy}(x, \eta_1),$$

arranging the numbers as given in (1), we have

$$f(x,y) = f(a,b) + (x-a) f_x(a,b) + (y-b) f_y(a,b)$$

$$+ \frac{1}{2!} [(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b)$$

$$+ (y-b)^2 f_{yy}(a,b)] + \frac{1}{3!} [(x-a)^3 f_{xxx} f(\xi_1, b)$$

$$+ 3(x-a)^2 (y-b) f_{xxy}(\xi_2, b) + 3(x-a)(y-b)^2 f_{xyy}(\xi_3, b)$$

$$(y-b)^3 f_{yyy}(a, \eta_1)]$$

This is the Taylor's formula for $n = 2$. The expression

$$R_2 = \frac{1}{3!}[(x-a)^3 f_{xxx}(\xi_1, b) + 3(x-a)^2(y-b) f_{xxy}(\xi_2, b)$$

$$+3(x-a)(y-b)^2 f_{xyy}(\xi_3, b) + (y-b)^3 f_{yyy}(a, \eta_1)].$$

This is called the remainder. If we denote $x - a = \Delta x$, $y - b = \Delta y$, and $\Delta\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}$, R_2 becomes

$$R_2 = \frac{1}{3!}[\frac{\Delta x^3}{\Delta\rho^3} f_{xxx}(\xi_1, b) + 3\frac{\Delta x^2 \Delta y}{\Delta\rho^3} f_{xxy}(\xi_2, b)$$

$$+3\frac{\Delta x \Delta y^2}{\Delta\rho^3} f_{xyy}(\xi_3, b) + \frac{\Delta y^3}{\Delta\rho^3} f_{yyy}(a, \eta_1)]\Delta\rho^3.$$

Example 9.1: Find the remainder R_2 of the function given by

$$f(x, y) = \sin x \sin y \ \textit{about} \ (0,0)$$

Solution:

$$f(x, y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)]$$

$$+\frac{1}{2!}[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_y y(0,0)] + R_2.$$

Where R_2 is given by

$$R_2 = \frac{1}{3!}[(x-a)^3 f_{xxx} f(\xi_1, b) + 3(x-a)^2 (y-b) f_{xxy}(\xi_2, b)$$

$$+3(x-a)(y-b)^2 f_{xyy}(\xi_3, b) + (y-b)^3 f_{yyy}(a, \eta_1)].$$

$$f_x(x,y) = \cos x \sin y, f_y(x,y) = \sin x \cos y,$$

$$f_{xx}(x,y) = -\sin x \sin y, f_{xy}(x,y) = \cos x \cos y,$$

$$f_{yy}(x,y) = -\sin x \sin y, f_{yx}(x,y) = \cos x \cos y,$$

$$f_{xxx} = -\cos x \sin y, f_{xxy} = -\sin x \cos y,$$

$$f_{xyy} = -\cos x \sin y, f_{yyy} = -\sin x \cos y$$

$$R_2 = \frac{1}{3!}[0 + 3x^2 y(-\sin(x + \theta_3 x))]$$

$$= -\frac{1}{2}[x^2 y \sin(x + \theta_3 x)]$$

Questions: Answer the following question.

1. Expand $z = \sin x \sin y$ in powers of $(x - \frac{\pi}{4})$ and $(y - \frac{\pi}{4})$. Find the terms of the first and second orders and R_2 (the remainder of second order).

2. Let $f(x, y) = e^x \sin y$. Expand $f(x + h, y + k)$ in powers of h and k and also find R_2.

3. Expand $x^2y + \sin y + e^x$ in powers of $(x-1)$ and $(y-\pi)$ through quadratic terms and write the remainder.

4. Expand $x^3 - 2xy^2$ in Taylor's Theorem about $a = 1,\ b = -1$.

5. Show that for $0 < \theta < 1$,

$$e^{ax}\sin by = by + abxy + \frac{1}{6}[(a^3x^3 - 3ab^2xy^2)\sin(b\theta y) + (3a^2bx^2y - b^3y^3)\cos(b\theta y)]e^{a\theta x}.$$

Keywords: Taylor's polynomial

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Suggested Readings

Tom M. Apostol (2003). Calculus, Volume II Second Editions, Publishers,John Willey & Sons, Singapore.

Lesson 10

Maximum and Minimum of function of two variables

10.1 Introduction

We say that a function $z = f(x,y)$ has a maximum (local) at a point (x_0,y_0) if

$$f(x_0,y_0) \geq f(x,y)$$

for all points (x,y) sufficiently close to the point (x_0,y_0).

A function of two variables has a absolute maximum (global maximum) at a point (x_0,y_0) if $f(x_0,y_0) \geq f(x,y)$ for all points (x,y) on the domain of the function.

Analogously we say that a function $z = f(x,y)$ has a minimum (local) at a point (x_0,y_0) if

$$f(x_0,y_0) \leq f(x,y)$$

for all points (x,y) sufficiently close to the point (x_0,y_0). Similarly we define absolute minimum (global minimum).

The maximum and minimum of a function are called extrema of the function; we say that a function has an extremum of a given point if it has a maximum or minimum at the given points.

Example 10.1. The function $z = (x-1)^2 + (y-2)^2 - 1$ contains a minimum at $x = 1, y = 2$.

Solution: As $f(1,2) = -1 < f(x,y)$ for all $x \neq 1$ and $y \neq 1$ i.e., $f(x,y) > f(1,2) = -1$

Example 10.2 The function $z = \frac{1}{2} - \sin(x^2 + y^2)$

Solution:

For $x = 0, \vec{y} \cong 0, f(0,0) = \frac{1}{2}$. Now for $0 < x^2 + y^2 < \frac{\pi}{6}$, $\sin(x^2 + y^2) > 0$.

So $f(0,0) > f(x,y)$, $0 < x^2 + y^2 < \frac{\pi}{6}$. i.e., $x = 0, y = 0$ is a maximum point of z.

Necessary Conditions for an Extremun: If a function $z = f(x,y)$ attains an extremum at $x = x_0$ and $y = y_0$, then each first partial derivative $(f_x, f_y)|_{(x_0,y_0)}$ either vanishes for these values or does not exist.

This result is not sufficient for investigating the extreme points, but permits finding these values for cases in which we are sure of the existence of a maximum or minimum. Otherwise more investigation is required.

Example 10.3. Consider the function $z = x^2 - y^2$

Solution:

The function has partial derivatives as $\frac{\partial z}{\partial x} = 2x$, $\frac{\partial z}{\partial y} = -2y$ which vanish at $x = 0$ and $y = 0$. But this function has neither maximum nor minimum at $x = 0$ and $y = 0$, since it takes both negative and positive values. Points at which $\frac{\partial z}{\partial x} = 0$ (or does not exist) $\frac{\partial z}{\partial y} = 0$ (or does not exist) are called critical points of the function $z = f(x, y)$. Thus if a function has an extreme point this can occur at the critical point. Converse may not true.

For investigation of a function at critical points, let us establish sufficient conditions for the maximum of a function of two variables, which can be generalized to functions of more than two variables also.

Theorem 10.1: Let a function f have continuous second partial derivatives on an open region containing a point (a, b) for which $f_x|_{(a,b)} = 0$ and $f_y|_{(a,b)} = 0$.

Let

$$d = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

or

$$d = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}_{(a,b)},$$

[$f_{xy} = f_{yx}$ as f has 2nd order continous partial derivatives]

1. If $d > 0$ and $f_{xx}(a,b) > 0$, then f has a local minimum at (a,b).

2. If $d > 0$ and $f_{xx}(a,b) < 0$, then f has a local maximum at (a,b).

3. If $d < 0$, then f has neither a local minimum nor a local maximum at (a,b).

4. The test is inconclusive if $d = 0$. (Additional investigation is required)

Proof follows from Taylor's theorem.

Note that if $d > 0$, then $f_{xx}(a,b)$ and $f_{yy}(a,b)$ must have same sign. This means that $f_{xx}(a,b)$ can be replaced by $f_{yy}(a,b)$.

Example 10.4 Find the extreme point of

$$f(x,y) = -x^3 + 4xy - 2y^2 + 1$$

Solution:

$$\frac{\partial f}{\partial x} = -3x^2 + 4y = 0, \frac{\partial f}{\partial y} = 4x - 4y = 0,$$

solving we obtain $x = y$. i.e., $3x^2 - 4x = 0$ or $x(3x-4) = 0$. So $(0,0)$ and $(\frac{4}{3}, \frac{4}{3})$ are the critical points. $f_{xx} = -6x$, $f_{yy} = -4$, $f_{xy} = 4$.

$$d = f_{xx}(0,0)f_{yy}(0,0) - [f_{xy}(0,0)]^2 = 0 - 16 < 0.$$

i.e., f has neither minimum nor maximum at critical point $(0,0)$. Hence $(0,0)$ is a saddle point. We will consider the critical point $(\frac{4}{3}, \frac{4}{3})$

$$d = f_{xx}(\tfrac{4}{3}, \tfrac{4}{3})f_{yy}(\tfrac{4}{3}, \tfrac{4}{3}) - [f_{xy}(\tfrac{4}{3}, \tfrac{4}{3})]^2$$

$$= -\frac{24}{3}(-4) - 16 = 16 > 0,$$

and $f_{xx}(\frac{4}{3}, \frac{4}{3}) = -8 < 0$, we conclude that $f(x,y)$ has a maximum at $(\frac{4}{3}, \frac{4}{3})$

References

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Suggested Readings

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Lesson 11

Lagrange's Multiplier Rule / Constrained Optimization

11.1 Introduction

We presents an introduction to optimization problems that involve finding a maximum or a minimum value of an objective function $f(x,y)$ subject to a

constraint of the form $g(x,y) = k$.

Maximum and Minimum. Finding optimum values of the function $f(x,y)$

without a constraint is a well known problem in calculus. One would normally use the gradient to find critical points (gradient (∇f) vanishes). Then check all stationary and boundary points to find optimum values.

Example 1.$\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$

$f(x,y) = 2x^2 + y^2, f_x(x,y) = 4x = 0, f_y(x,y) = 2y = 0, f(x,y)$

has a critical/ stationary point at (0,0).

The Hessian: A common method of determining whether or not a function has an extreme value at a stationary point is to evaluate the hessian of the function of n variables at that point. where the hessian is defined as

$$H(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial {x_1}^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial {x_2}^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial {x_n}^2} \end{pmatrix}$$

A square matrix of order $n \times n$ is said to be positive definite if its leading principal minors are all positive.

For n=2, we have

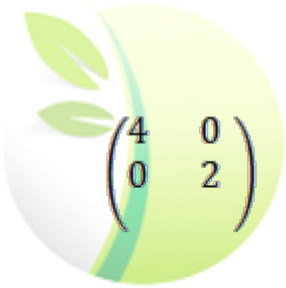
$$H(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

Second Derivative Test: The Second derivative test determines the optimality of stationary point x according to the following rules:

Let $\frac{\partial^2 f}{\partial x^2} = A, \frac{\partial^2 f}{\partial x \partial y} = B, \frac{\partial^2 f}{\partial y^2} = C$, and $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ at the point (x, y), then

1. If $A>0$ and $AC-B^2>0$ at the point (x,y), then f has a local minimum at (x,y).
2. If $A<0$ and $AC-B^2>0$ at the point (x,y), then f has a local maximum at (x,y).
3. If $AC-B^2<0$ at (x,y), then (x,y) is a saddle point of f.
4. If $AC-B^2=0$, further investigation is required.

In the above Example 1,



Therefore $f(x,y)$ has a minimum at (0,0) as $4>0$ and determinant of the matrix is $8>0$.

11.1.1 Constrained Maximum and Minimum

When finding the extreme values of $f(x,y)$ subject to a constraint $g(x,y)=k$, the

stationary points found above will not work. This new problem can be thought of as finding extreme values of $f(x,y)$ when the point (x,y) is restricted to lie on the

surface $g(x,y) = k$. The value of $f(x,y)$ is maximized (minimized) when the surfaces touch each other,i.e , they have a common tangent for line.

This means that the surfaces, gradient vectors at that point are parallel, hence,

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$

The number λ in the equation is known as the Lagrange multiplier.

11.2 Lagrange multiplier method

The Lagrange multiplier methods solves the constrained optimization problem by transforming it into a non-constrained optimization problem of the form:

$$L(x,y,\lambda) = f(x,y) + \lambda(k - g(x,y))$$

or $(g - k)$). Then finding the gradient and Hessian as was done above will determine any optimum values of $L(x,y,\lambda)$.

Suppose we want to find optimum values for the following:

Example 11.2: $f(x,y) = 2x^2 + y^2$ subject to $x + y = 1$.

Then the Lagrangian method will result in a non-constrained function.

$L(x,y,\lambda) = 2x^2 + y^2 + \lambda(1 - x - y)$. The gradient for this new function is

$$\frac{\partial L}{\partial x} = 4x - \lambda = 0$$

$$\frac{\partial L}{\partial y} = 2y - \lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = 1 - x - y = 0$$

Solving x, y, λ, we obtain $x = \frac{1}{3}$, $y = \frac{2}{3}$ and $\lambda = \frac{4}{3}$.

The Hessian matrix at the stationary point

$$H(L) = \begin{pmatrix} 4 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 0 \end{pmatrix}$$

Since $H(L)$ is positive definite the solution $x = \frac{1}{3}$, $y = \frac{2}{3}$ minimizes

$f(x,y) = 2x^2 + y^2$ subject to $x + y = 1$ with $f(\frac{1}{3}, \frac{2}{3}) = \frac{2}{3}$

Example 11.3: Find the rectangle of parameter *l* which has maximum area i.e., Maximize xy subject to

$$2(x+y) = l$$

Solution:

$$L(x,y,\lambda) = xy + \lambda 2(x+y) - l).$$

$$\frac{\partial L}{\partial x} = y + 2\lambda = 0$$

$$\frac{\partial L}{\partial y} = x + 2\lambda = 0$$

i.e., $x = y = -2\lambda$ i.e., $-8\lambda = l \Rightarrow \lambda = -\frac{l}{8}$.

$x = y = \frac{l}{4}$, so that the rectangle of maximum area is a square.

Example 11.4 Find the shortest distance from the point (1,0) to the parabola $y^2 = 4x$, i.e., Minimize $(x-1)^2 + y^2$ subject to $y^2 = 4x$.

$$L(x,y,\lambda) = (x-1)^2 + y^2 + \lambda(y^2 - 4x)$$

$$\frac{\partial L}{\partial x} = 2(x-1) - 4\lambda = 0, \frac{\partial L}{\partial y} = 2y + 2\lambda y = 0$$

$$y^2 - 4x = 0$$

Now $2y + 2\lambda y = 0 \Rightarrow y = 0 \; or \; \lambda = -1$

If $\lambda = -1,$ then $x = -1$, from $2(x-1) - 4\lambda = 0$

Hence $y = 0 \Rightarrow x = 0$

$x = -1$

Now $y^2 = 4^n, y^2 = -4,$ so $y = \sqrt{-4}$ not possible no real value.

Hence $y = 0, x = 0,$

i.e., $\lambda = -\frac{1}{2}$

Hence the only solution is $x = 0, y = 0, \lambda = -\frac{1}{2}$ and the required distance is unity.

Questions: Answer the following question

1. Determine the maximum value of the n-th root of a product of numbers $x_1, x_2, \cdots, x_n$ provided that their sum is equal to a given number a. Thus the problem is stated as follows: it is required to find the maximum of the function $z = \sqrt[n]{x_1 . x_2 \cdots, x_n}$ subject to $\sum_{i=1}^{n} x_i - a = 0$, $x_i > 0$, for all i.

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Suggested Readings

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Lesson 12

Convexity, Concavity and Points of Inflexion

12.1 Introduction

In the plane, we consider a curve $y = f(x)$, which is the graph of a single-valued differentiable function $f(x)$.

Definition 12.1: We say that the curve is convex downward bending up on the interval (b, c) if all points of the curve lie above the tangent at any point on the interval. Or when the curve turns anti-clock wise we call it is convex downward (concave upward) (see Fig. 1).

Fig.1. (Convex downward/Bending up)

Definition: We say that a curve is convex upwards for bending down on the interval (a, b) if all points of the curve lie below the tangent at any point on the interval. Or when the curve turns clock-wise we say it is convex upward (concave downward) (see Fig. 2).

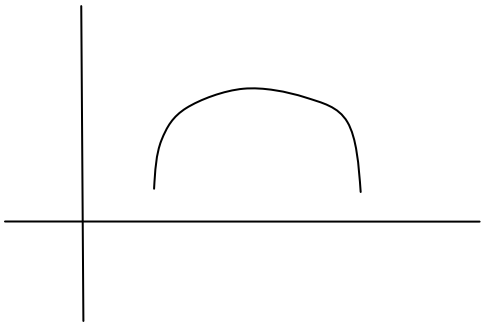


Fig. 2. (Convex upward / Bending down)

The curve has a point of inflexion at P, at which the curve changes from convex upwards to convex downwards and vice-versa.

Theorem 1: If for all points of an interval (a, b), $f''(x) < 0$, the curve $y = f(x)$ on this interval is convex upward. If $f''(x) > 0$, the curve is convex downward.

If $f''(x) < 0 \; \forall \; x \in (a, b) \Rightarrow y = f(x)$ is convex upward on (a, b).

If $f''(x) > 0 \; \forall \; x \in (a, b) \Rightarrow y = f(x)$ is convex doward on (a, b).

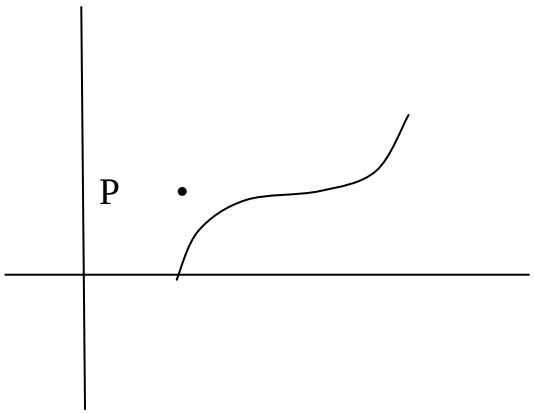
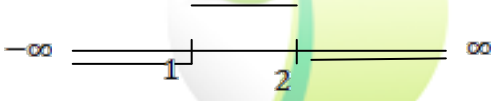


Fig. 3. (Inflexion point)

Example 12.1: Find the ranges of values of x for which the curve $y = x^4 - 6x^3 + 12x^2 + 5x + 7$ is convex downwards, convex upwards, and also determine the point of inflection.

Solution:

$$y' = 4x^3 - 18x^2 + 24x + 5,$$



$$y'' = 12x^2 - 36x + 24 = 12(x-1)(x-2)$$

Now on the interval $(-\infty, 1)$, $x - 1 < 0$, $x - 2 < 0$, hence $y'' > 0$. If $x > 2$, $x > 1$, i.e., $x - 2 > 0$ and $x - 1 > 0$. Hence for $x \in (2, \infty)$, $y'' > 0$. Now on the interval $(1,2)$, $y'' < 0$. Hence the curve is convex downward on the interval $(-\infty, 1)$ and $(2, \infty)$. Convex upwards on $(1,2)$. The curve has inflection points at $= 1$ and $x = 2$ as y'' changes sign. At $x = 1$, $y = 19$ and at $x = 2$, $y = 33$. i.e., $(1,19)$ and $(2,33)$ are two points of inflection of the curve.

Example 12.2: Determine the intervals where the graph of the function is convex downward and convex upward of $f(x) = \frac{x}{2x-1}$

Solution:

$$f(x) = [(x - \tfrac{1}{2}) + \tfrac{1}{2}]/[2(x - \tfrac{1}{2})] = \tfrac{1}{2}\{1 + [1/(2x-1)]\}.$$

Hence,

$$f'(x) = \tfrac{1}{2}[-1/(2x-1)^2].2 = -\frac{1}{(2x-1)^2}.$$

Then $f''(x) = 4/(2x-1)^3$. For $x > \frac{1}{2}$, $2x - 1 > 0$, $f''(x) > 0$, the graph is convex downward. For $x < \frac{1}{2}$, $2x - 1 < 0$, $f''(x) < 0$, the graph is convex upward. There is no inflection point, since $f(x)$ is not defined when $x = 1/2$.

Example 12.3: Determine the intervals where the graph of the function is convex downward and convex upward of $f(x) = 5x^4 - x^5$,

Solution:

$f'(x) = 20x^3 - 5x^4$, and $f''(x) = 60x^2 - 20x^3 = 20x^2(3 - x)$. So, for $0 < x < 3$ and for $x < 0$, $3 - x > 0$, $f''(x) > 0$, the graph is convex downward. For $x > 3$, $3 - x < 0$, $f''(x) < 0$, and the graph is convex upward.

There is an inflection point at $(3,162)$. There is no inflection point at $x = 0$, the graph is convex downward for $x < 3$.

Example 12.4: Find the point of inflection of the curve $y = (\ln x)^3$,

Solution:

$y'(x) = 3(\ln x)^2 . \frac{1}{x}$, $y'' = \frac{3\ln x}{x^2}(2 - \ln x)$. $y'' = 0$ if $\ln x = 0$, or $\ln x = 2$. i.e., $x = 1$ or $x = e^2$. Now y'' changes sign from negative to positive as x passes through 1 and changes sign from positive to negative as x passes through e^2. Thus $(1,0)$ and $(e^2,8)$ are two points of inflection of the given curve.

Example 12.5: What conditions must the coefficients a, b, c satisfy for the curve $y = ax^4 + bx^3 + cx^2 + dx + e$ to have points of inflection?

Solution:

$y'' = 12ax^2 + 6bx + 2c$ has a point of inflection iff the equation $2ax^2 + 6bx + 2c = 0$ has different real roots. i.e., discriminant $D = 9b^2 - 24ac > 0$ is positive. i.e. $3b^2 > 8ac$.

Questions: Answer the following questions.

1. Determine all the inflexion points of $sin\, x$.

2. Determine all the inflexion points of $cos\, x$.

3. Determine all the inflexion points of $f(x) = \tan x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$

4. Sketch the curve $y = sin^2 x$. Determine the inflexion points. Compare with graph of $|\sin x|$.

5. Determine the inflexion points and the intervals of convex downward / bending up and convex upward / bending down for the following curve

6. $y = x + \frac{1}{x}$

7. $y = \frac{x}{x^2+1}$

8. $y = \frac{x}{x^2-1}$

9. Sketch the curve $y = \frac{1}{6}(x^3 - 6x^2 + 9x + 6)$

10. Pint of inflexion of $y = x^4$.

Keywords: Convex up, Convex down, Inflexion Point.

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Suggested Readings

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Lesson 13

Curvature

13.1 Introduction

Curvature measures the extent to which a curve is not contained in a straight line. It curvature measures how curved the curve is. We have heard the comparison of bending or curvature of a road at two of its points. The curvature of a straight line is zero. It also measures how fast the tangent vector turns as a point moves along the curve.

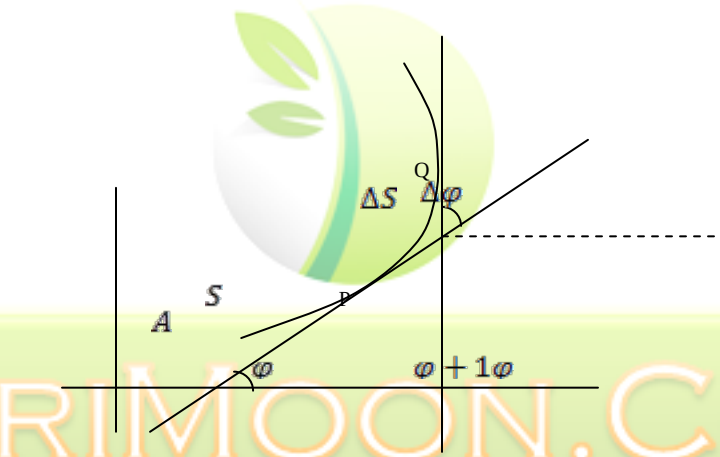

Fig.1.

Let A be a fixed point on the curve. Let arc $AP = s$, and arc $AQ = s + \Delta s$, so that arc $PQ = \Delta s$. Let ϕ, $\phi + \Delta\phi$ be the angles which the tangents at P and Q make with some fixed line (say x- axis). $\Delta\phi$ denotes the angle formed by these tangents. The symbol $\Delta\phi$ also denotes the angle through which the tangent turns from P and Q through a distance Δs. $\Delta\phi$ will be large or small, as compared with Δs, depending the degree of the sharpness of the bend. This suggests the following definitions:

The curvature of the curve at P is defined as $\lim_{Q\to P}|\frac{\Delta\phi}{\Delta s}| = |\frac{d\phi}{ds}|$.

The reciprocal of curvature $\rho = \frac{ds}{d\phi}$ is the radius of curvature.

Length of Arc as a Function, Derivative of Arc.

Let $y = f(x)$ be the equation of a given curve on which we take a fixed point A. Let $P(x, y)$ and $Q(x + \Delta x, y + \Delta y)$ be the variable points on the curve with arc $AP = s$ and arc $AQ = s + \Delta s$ so that arc $PQ = \Delta s$.

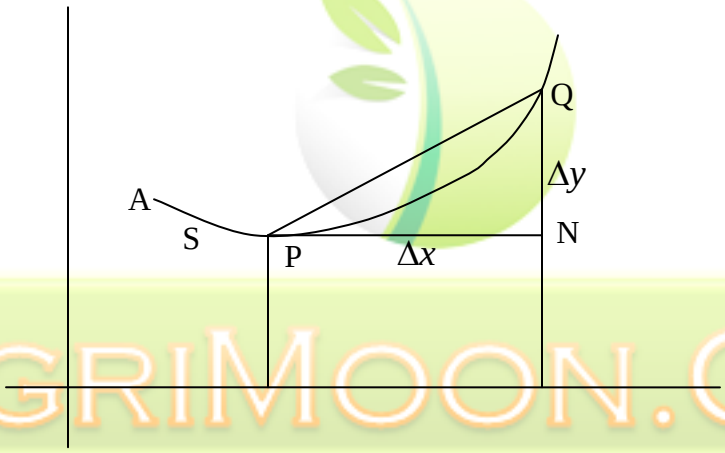

Fig. 2.

$$chord PQ^2 = PN^2 + NQ^2 = \Delta x^2 + \Delta y^2$$

$\Rightarrow$

$$(\frac{chord\ PQ}{\Delta x})^2 = 1 + (\frac{\Delta y}{\Delta x})^2$$

$\Rightarrow$

$$[\frac{chord\ PQ}{arc\ PQ}]^2(\frac{\Delta s}{\Delta x})^2 = 1 + (\frac{\Delta y}{\Delta x})^2$$

$\lim_{Q\to P}\frac{chord\ PQ}{arc\ PQ} = 1$, taking limit $\lim_{Q\to P}$ both sides we have

$$(\frac{ds}{dx})^2 = 1 + (\frac{dy}{dx})^2$$

$\Rightarrow$

$$\frac{ds}{dx} = \sqrt{1+(\frac{dy}{dx})^2}$$

Radius of Curvature: Cartesian Equations

We define the absolute value of $\frac{d\phi}{ds}$ as the curvature and denote it by $\kappa = |\frac{d\phi}{ds}|$.

Consider the curve $y = f(x)$, we note that $\tan\phi = \frac{dy}{dx}$ and, therefore,

$$\phi = \tan^{-1}(\frac{dy}{dx})$$

Differentiating this with respect to x, we have

$$\frac{d\phi}{dx} = \frac{\frac{d^2y}{dx^2}}{1+(\frac{dy}{dx})^2}$$

As $\frac{ds}{dx} = \sqrt{1+(\frac{dy}{dx})^2}$, we have

$$\frac{d\phi}{ds} = \frac{\frac{d\phi}{dx}}{\frac{ds}{dx}} = \frac{\frac{\frac{d^2y}{dx^2}}{1+(\frac{dy}{dx})^2}}{\sqrt{1+(\frac{dy}{dx})^2}} = \frac{\frac{d^2y}{dx^2}}{[1+(\frac{dy}{dx})^2]^{\frac{3}{2}}}$$

Hence $\rho = |\frac{ds}{d\phi}| = \frac{(1+y_1{}^2)^{\frac{3}{2}}}{y_2}$, where $y_1 = \frac{dy}{dx}$, $y_2 = \frac{d^2y}{dx^2}$

Note: If $\rho = \frac{ds}{d\phi}$, the radius of curvature, ρ, is positive or negative according as $\frac{d^2y}{dx^2}$ is +ve or -ve i.e., accordingly as the curve is convex downward or convex upward. But we consider ρ is +ve here. Curvature is zero at point of inflection. Since ρ is independent of the choice of x-axis and y-axis, interchanging x and y, we see that ρ, is given by

$$\frac{[1+(\frac{dx}{dy})^2]^{\frac{3}{2}}}{|\frac{d^2x}{dy^2}|}.$$

Curvature- parametric Equation

Given $x = f(t)$, $y = F(t)$. $f'(t) \neq 0$.

$$\frac{dy}{dx} = \frac{F'(t)}{f'(t)}, \frac{d^2y}{dx^2} = \frac{f'F'' - F'f''}{[f']^3}$$

Hence the curvature $\kappa = \frac{|f'F'' - F'f''|}{[(f')^2+(F')^2]^{\frac{3}{2}}}$ $\kappa = \frac{ds}{d\phi}$

Curvature- polar Equation

Let $r = f(\theta)$ be the given curve in polar co-ordinates. Now its cartesian coordinates are of the form $x = r\cos\theta$, $y = r\sin\theta$. i.e., $x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$. Now

$$\frac{dx}{d\theta} = \frac{df}{d\theta}\cos\theta - f(\theta)\sin\theta = \frac{dr}{d\theta}\cos\theta - r\sin\theta$$

and

$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

$$\frac{d^2x}{d\theta^2} = \frac{d^2r}{d\theta^2}\cos\theta - 2\frac{dr}{d\theta}\sin\theta - r\cos\theta$$

$$\frac{d^2y}{d\theta^2} = \frac{d^2r}{d\theta^2}\sin\theta + 2\frac{dr}{d\theta}\cos\theta - r\sin\theta$$

substituting the latter expressions in the previous parametric-form, we have

$$\kappa = \frac{|r^2+2r'^2-rr''|}{(r^2+r'^2)^{\frac{3}{2}}}(*)$$

We know

$$\kappa = \frac{|f'F'' - F'f''|}{[(f')^2 + (F')^2]^{\frac{3}{2}}}$$

numerator becomes

$$(r''\sin\theta + 2r'\cos\theta - r\sin\theta) \times (r'\cos\theta - r\sin\theta)$$

$$-(r'\sin\theta + r\cos\theta) \times (r''\cos\theta - 2r'\sin\theta - r\cos\theta)$$

$$= r''r'\sin\theta\cos\theta + 2r'^2\cos^2\theta - rr'\sin\theta\cos\theta$$

$$-rr''\sin^2\theta - 2rr'\sin\theta\cos\theta + r^2\sin^2\theta$$

$$-r'r''\sin\theta\cos\theta + 2r'^2\sin^2\theta + rr'\sin\theta\cos\theta$$

$$-rr''\cos^2\theta + 2rr'\sin\theta\cos\theta + r^2\cos^2\theta$$

$$= r^2(sin^2\theta + \cos^2\theta) + 2r'^2(\cos^2\theta + \sin^2\theta)$$

$$-rr''(sin^2\theta + \cos^2\theta)$$

To check we can observe that

$$[f'F'' - F'f''] = |r^2 + 2r'^2 - rr''|$$

denominator becomes

$$(r'\cos\theta - r\sin\theta)^2 + (r'\sin\theta + r\cos\theta)^2$$

$$= r'^2\cos^2\theta + r^2\sin^2\theta - 2rr'\sin\theta\cos\theta$$

$$r'^2\sin^2\theta + r^2\cos^2\theta + 2rr'\sin\theta\cos\theta$$

$$= r'^2(\cos^2\theta + \sin^2\theta) + r^2(sin^2\theta + \cos^2\theta)$$

$$[(f')^2 + (F')^2]^{\frac{3}{2}} = \left(r'^2 + r^2\right)^{\frac{3}{2}}$$

Hence

$$K = \frac{|r^2+2r'^2-rr''|}{(r^2+r'^2)^{\frac{3}{2}}}$$

The radius of curvature is

$$\rho = \frac{(r^2+r'^2)^{\frac{3}{2}}}{|r^2+2r'^2-rr''|}$$

Example 1: Determine the radius of curvature of the curve $r = a\theta\ (a > 0)$

Solution:

$\frac{dr}{d\theta} = a, \frac{d^2r}{d\theta^2} = 0$

Hence

$$\rho = \frac{(a^2\theta^2+a^2)^{\frac{3}{2}}}{a^2\theta^2+2a^2} = \frac{a(\theta^2+1)^{\frac{3}{2}}}{\theta^2+2}$$

* We know

$$K = \frac{|f'F''-F'f''|}{[(f')^2+(F')^2]^{\frac{3}{2}}}$$

numerator becomes

$$(r''\sin\theta + 2r'\cos\theta - r\sin\theta) \times (r'\cos\theta - r\sin\theta)$$

$$-(r'\sin\theta + r\cos\theta) \times (r''\cos\theta - 2r'\sin\theta - r\cos\theta)$$

$$= r''r'\sin\theta\cos\theta + 2r'^2\cos^2\theta - rr'\sin\theta\cos\theta$$

$$-rr''\sin^2\theta - 2rr'\sin\theta\cos\theta + r^2\sin^2\theta$$

$$-r'r''\sin\theta\cos\theta + 2r'^2\sin^2\theta + rr'\sin\theta\cos\theta$$

$$-rr''\cos^2\theta + 2rr'\sin\theta\cos\theta + r^2\cos^2\theta$$

$$= r^2(sin^2\theta + \cos^2\theta) + 2r'^2(\cos^2\theta + \sin^2\theta)$$

$$-rr''(sin^2\theta + \cos^2\theta)$$

$$= |r^2 + 2r'^2 - rr''|$$

denominator becomes

$$(r'\cos\theta - r\sin\theta)^2 + (r'\sin\theta + r\cos\theta)^2$$

$$= r'^2\cos^2\theta + r^2\sin^2\theta - 2rr'\sin\theta\cos\theta$$

$$r'^2\sin^2\theta + r^2\cos^2\theta + 2rr'\sin\theta\cos\theta$$

$$= r'^2(\cos^2\theta + \sin^2\theta) + r^2(sin^2\theta + \cos^2\theta)$$

$$r'^2 + r^2$$

Hence

$$K = \frac{|r^2+2r'^2-rr''|}{(r^2+r'^2)^{\frac{3}{2}}}(*)$$

Example 2: Find the radius of curvature of $r = a\sec^2\frac{\theta}{2}$

Ans.: $\rho = 2a\sec^3\frac{\theta}{2}$

Example : Find the radius of curvature of $x = 3t^2$, $y = 3t - t^3$ for $t = 1$,

Ans.: $\rho = 6$

Example : Find the curvature of the hyperbola $xy = 1$ at $(1,1)$.

Solution:

$y' = -\frac{1}{x^2}$ and $y'' = \frac{2}{x^3}$. So

$$\kappa = \frac{\frac{2}{x^3}}{[1+(\frac{1}{x^4})]^{\frac{3}{2}}} = \frac{2}{x^3}\frac{x^6}{(x^4+1)^{\frac{3}{2}}} = \frac{2x^3}{(x^4+1)^{\frac{3}{2}}}$$

When $x = 1$, $\kappa = \frac{2}{2\sqrt{2}} = \frac{\sqrt{2}}{2}$.

Example 3: For what value of x is the radius of curvature of $y = e^x$ smallest?

Solution:

$y' = y'' = e^x$, $\kappa = \frac{e^x}{(1+e^{2x})^{\frac{3}{2}}}$ and radius of curvature ρ is $\frac{(1+e^{2x})^{\frac{3}{2}}}{e^x}$. Then

$$\frac{d\rho}{dx} = \frac{e^x \cdot \frac{3}{2}(1+e^{2x})^{\frac{1}{2}}(2e^{2x}) - e^x(1+e^{2x})^{\frac{3}{2}}}{e^{2x}}$$

$$= \frac{(1+e^{2x})^{\frac{1}{2}}[3e^{2x} - (1+e^{2x})]}{e^x}$$

$$= \frac{(1+e^{2x})^{\frac{1}{2}}(2e^{2x}-1)}{e^x}$$

etting $\frac{d\rho}{dx} = 0$, we find $2e^{2x} = 1$, $2x = \ln\frac{1}{2} = -\ln 2$, $x = -\frac{(\ln 2)}{2}$. As the second derivative at this point is positive, $x = -\frac{(\ln 2)}{2}$ is the point which gives the smallest radius of curvature.

Example 4: Find the radius of curvature at any point on the curves: $y = c\cosh\frac{x}{c}$

Solution:

$y' = c\sinh\frac{x}{c}.\frac{1}{c} = \sinh\frac{x}{c}$, $y'' = \frac{1}{c}\cosh\frac{x}{c}$

$$\rho = \frac{[1+(\frac{dy}{dx})^2]^{\frac{3}{2}}}{|\frac{d^2y}{dx^2}|} = \frac{[1+\sinh^2\frac{x}{c}]^{\frac{3}{2}}}{\frac{1}{c}\cosh\frac{x}{c}}$$

$$= \frac{(\cosh^2\frac{x}{c})^{\frac{3}{2}}}{\frac{1}{c}\cosh\frac{x}{c}} = c\cosh^2\frac{x}{c}$$

$$y^2 = c^2\cosh^2\frac{x}{c}$$

implies

$$\frac{y^2}{c} = c\cosh^2\frac{x}{c}$$

i.e. $\rho = \frac{y^2}{c}$.

Example : Find the radius of curvature at the origin of the curve

$$y - x = x^2 + 2xy + y^2$$

Solution:

$$\frac{dy}{dx} - 1 = 2x + 2x\frac{dy}{dx} + 2y + 2y\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}|_{(0,0)} = 1$$

$$\frac{d^2y}{dx^2} = 2 + 2\frac{dy}{dx} + 2x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2(\frac{dy}{dx})^2 + 2y\frac{d^2y}{dx^2}$$

which implies $\frac{d^2y}{dx^2}|_{(0,0)} = 8$.

$$\rho = \frac{(1+y_1)^{\frac{3}{2}}}{y_2} = \frac{2^{\frac{3}{2}}}{8} = \frac{\sqrt{2}}{4}$$

Example 5: Find the curvature of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$ at an arbitrary point (x, y).

Solution:

$\frac{dx}{dt} = a(1-\cos t)$, $\frac{d^2x}{dt^2} = a\sin t$, $\frac{dy}{dt} = a\sin t$, $\frac{d^2y}{dt^2} = a\cos t$. Using this parametric formula $\kappa = \frac{|f'F''-F'f''|}{[(f')^2+(F')^2]^{\frac{3}{2}}}$, we obtain

$$\kappa = \frac{|a(1-\cos t)a\cos t - a\sin t.a\sin t|}{[a^2(1-\cos t)^2+a^2\sin^2 t]^{\frac{3}{2}}}$$

$$= \frac{|a^2(\cos t-\cos^2 t-\sin^2 t)|}{[2a^2(1-\cos t)]^{\frac{3}{2}}}$$

$$= \frac{|\cos t-1|}{2^{\frac{3}{2}}a(1-\cos t)^{\frac{3}{2}}}$$

$$= \frac{1}{2^{\frac{3}{2}}a(1-\cos t)^{\frac{1}{2}}} = \frac{1}{|4a\sin\frac{t}{2}|}$$

When $t = \pi$, $\kappa = \frac{1}{|4a|}$

Questions: Answer the following questions.

1. Find the curvature of the curve $b^2x^2 + a^2y^2 = a^2b^2$ at the point (a,b) and (a,0)

2. Find the curvature of the curve $16y^2 = 4x^4 - x^6$ at the point (2,0)

3. Find the curvatur e of the curve $xy = 12$ at the point (3,4)

Questions: Find the radius of curvature of the following curves at the indicated points.

4. $y = x^3$ at the point (4,8)

5. $x^2 = 4ay$ at the point $(0,0)$

6. $y = \ln x$ at the point (1,0)

7. $y = \sin x$ at the point $\left(\frac{\pi}{2}, 1\right)$

8. Find the point of the curve $y = e^x$ at which the radius of curvature is minimum.

Ans.: 1. $\frac{b}{a^2}, \frac{a}{b^2}$, 2. $\frac{1}{2}$, 3. $\frac{24}{125}$, 4. $\frac{80\sqrt{10}}{3}$, 5. 29, 6. $2\sqrt{2}$, 7. 1 & 8. $-\frac{1}{2}ln2, \frac{\sqrt{2}}{2}$

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Suggested Readings

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Lesson 14

Asymptotes

14.1 Introduction

A straight line d is called an asymptote to a curve C (fig.1), if the distance δ distance from a point P of C to d approaches to zero as P recedes to infinity.

Roughly speaking, a straight line is said to be an asymptote of a curve if it comes arbitrary close to that curve (but never touches the curve).

14.1.1 Asymptotes of Functions: If the graph of a function has an asymptote d, then we say that the function has an asymptote d. A function can have more than one asymptote. If an asymptote is parallel with the y-axis, we call it a vertical asymptote. If an asymptote is parallel with the x-axis,

we call it a horizontal asymptote. All other asymptotes are oblique asymptotes.

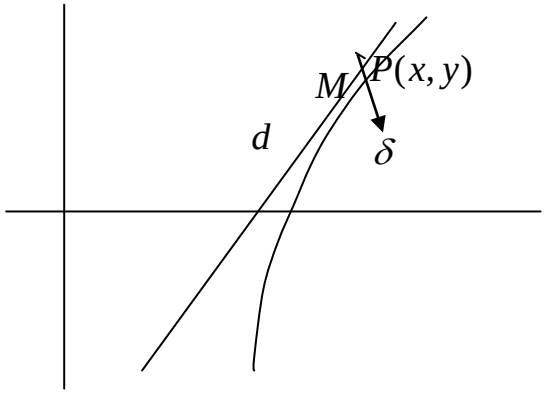

Fig. 1

Vertical Asymptotes

A straight line $x = a$ is a vertical asymptote to the the curve $y = f(x)$ if $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$. Consequently, to find vertical asymptotes one has to find values of $x = a$ such that when they are approached by the function $y = f(x)$, the latter approaches infinity. Then the straight line is a vertical asymptote.

Example 14.1: The curve $y = \frac{2}{x-5}$ has a vertical asymptote $x = 5$, since $y \to \infty$ as $x \to 5^+$.

Example 14.2: The curve $y = \tan x$ has infinite number of vertical asymptotes at $x = \frac{n\pi}{2}$ for $n = 1,3,5,\cdots$, as $\tan x \to \infty$ when $x \to \pm\frac{n\pi}{2}$.

Example 14.3: The curve $y = \frac{x^2+3x+2}{x+2}$ has no vertical asymptote at $x = -2$ as $\lim_{x\to -2} \frac{x^2+3x+2}{x+2} = -1$.

14.2 Horizontal Asymptotes

A line $y = b$ is a horizontal asymptote of a function $f(x)$ iff $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$, with $b \in R$.

Examples 14.4: The curve $y = \frac{3x^2-4x-1}{6x^2-6}$ has horizontal asymptote as $\lim_{x\to\infty} \frac{3x^2-4x-1}{6x^2-6} = \frac{1}{2}$. So, $y = \frac{1}{2}$ is a horizontal asymptote of the function $\frac{3x^2-4x-1}{6x^2-6}$.

14.3 Oblique Asymptotes/Inclined Asymptotes

Let the curve $y = f(x)$ have an inclined or oblique asymptote d(fig.1) whose equation is $y = mx + c$.

Here m and c are unknown real numbers to be determined. Let $PM = \delta$ be the perpendicular distance of any point $P(x, y)$ on the curve to the line $y = mx + c$.

Hence, $\delta = \frac{y - mx - c}{\sqrt{1+m^2}}$. Now $\delta \to 0$ as $x \to \infty$. Hence, $\lim_{x\to\infty}[y - mx - c] = 0$. i.e., $\lim_{x\to\infty}[y - mx] = c$, hence

$$\lim_{x\to\infty}\left[\frac{y}{x} - m\right] = \lim_{x\to\infty}[y - mx].\lim_{x\to\infty}\frac{1}{x}$$

$= c.0 = 0.$

So $m = \lim_{x\to\infty}\frac{y}{x}$.

Example 14.5: Find the asymptotes to the curve $y = \frac{x^2+2x-1}{x}$

Solution:

When $x \to 0^-$, $y \to +\infty$, and $x \to 0^+$, $y \to -\infty$, hence the straight line $x = 0$ is a vertical asymptote of the above curve.

Next to find the asymptotes of the form $y = mx + c$, i.e., the inclined asymptote.

$$m = \lim_{x\to\infty} \frac{y}{x} = \lim_{x\to\infty} \frac{x^2+2x-1}{x^2}$$

$$= \lim_{x\to\infty} [1 + \frac{2}{x} - \frac{1}{x^2}] = 1$$

$$c = \lim_{x\to\infty} [y - mx] = \lim_{x\to\infty} [y - x]$$

$$= [\frac{x^2+2x-1}{x} - x] = \lim_{x\to\infty} [2 - \frac{1}{x}] = 2.$$

Hence $y = x + 2$ is an inclined asymptotes to the given curve.

Example 14.6: Find the oblique asymptotes to the curve $y = \sqrt{x^2 - 1} + 2$

Solution:

$y = -x + 2$

14.3.1 Tutorial Discussion

- An asymptote is a straight line which acts as a boundary for the graph of a function.

- When a function has an asymptote (and not all functions have them) the function gets closer and closer to the asymptote as the input value to the function approaches either a specific value a or positive or negative infinity.
- The functions most likely to have asymptotes are rational functions
- Vertical asymptotes occur when the following condition is met:

The denominator of the simplified rational function is equal to 0.

Remember, the simplified rational function has cancelled any factors common to both the numerator and denominator.

e.g., Given the function $f(x) = \frac{2-5x}{2+2x}$

The first step is to cancel any factors common to both numerator and denominator. In this case there are none.

The second step is to see where the denominator of the simplified function equals 0. $2 + 2x = 0$ implies $x = -1$.

The vertical line $x = -1$ is the only vertical asymptote for the function. As the input value x to this function gets closer and closer to -1 the function itself looks and acts more and more like the vertical line $x = -1$.

Example 14.7 $f(x) = \frac{2x^2+10x+12}{x^2-9}$

First simplify the function. Factor both numerator and denominator and cancel any common factors.

$$f(x) = \frac{2x^2+10x+12}{x^2-9} = \frac{(x+3)(2x+4)}{(x+3)(x-3)} = \frac{2x+4}{x-3}$$

The asymptote(s) occur where the simplified denominator equals 0. i.e., $x - 3 = 0$.

The vertical line $x = 3$ is the only vertical asymptote for this function. As the input value x to this function gets closer and closer to 3 the function itself looks more and more like the vertical line $x = 3$.

Example 14.8 If $g(x) = \frac{x-5}{x^2-x-6}$

Factor both the numerator and denominator and cancel any common factors.
In this case there are no common factors to cancel.

$$\frac{x-5}{x^2-x-6} = \frac{x-5}{(x+2)(x-3)}$$

The denominator equals zero whenever either $x + 2 = 0$ or $x - 3 = 0$. Hence this function has two vertical asymptotes, one at $x = -2$ and the other at $x = 3$.

5. Horizontal Asymptotes

Horizontal asymptotes occur when either one of the following conditions is met (you should notice that both conditions cannot be true for the same function).

- The degree of the numerator is less than the degree of the denominator. In this case the asymptote is the horizontal line $y = 0$.
- The degree of the numerator is equal to the degree of the denominator. In this case the asymptote is the horizontal line $y = \frac{a}{b}$ where a is the leading coefficient in the numerator and b is the leading coefficient in the denominator.

When the degree of the numerator is greater than the degree of the denominator there is no horizontal asymptote.

Example 14.9 $f(x) = \frac{x^2-3x+5}{x^3-27}$

then there is a horizontal asymptote at the line $y = 0$ because the degree of the numerator 2 is less than the degree of the denominator 3.

This means that as x gets larger and larger in both the positive and negative directions ($x \to \infty$) and ($x \to -\infty$) the function itself looks more and more like the horizontal line $y = 0$

Find the vertical asymptotes, horizontal asymptotes and inclined asymptotes for each of the following functions Problems:

Exercises:

Find the asymptotes of the following curves:

1. $f(x) = \frac{x^2+2x-15}{x^2+7x+10}$

Solution: Vertical: $x = -2$ Horizontal: $y = 1$ Inclined: none

2. $g(x) = \frac{2x^2-5x+7}{x-3}$

Solution: Vertical: $x = 3$ Horizontal: none Inclined: $y = 2x + 1$

3. $y = \frac{x^2+1}{1+x}$ Ans. $x = -1, y = x - 1$

4. $y = x + e^{-x}$ Ans. $y = x$

5. $y = a^3 - x^2$ Ans. No asymptotes

6. $y = x \ln(e + \frac{1}{x})$ Ans. $x = -\frac{1}{e}, y = x + \frac{1}{e}$

7. $y = x\, e^{\frac{1}{x^2}}$ Ans. $x = 0$

8. Sketch the function $y = \frac{x^2-3}{2x-4}$

Keywords: Asymptotes, horizantal, vertical and inclied asymptotes.

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Suggested Readings

Tom M. Apostol, (2003). Calculus, Volume II Second Editions, Publishers,John Willey & Sons, Singapore.

Lesson 15

Tracing of Curves

15.1 Introduction

Now we use some mathematical techniques to trace curves and graphs of functions much more efficiently. We shall especially look for the following aspects of the curve.

1. Intersection with the coordinate axes.

2. Critical points

3. Regions of increase

4. Regions of decrease

5. Maxima and minima (including local ones)

6. Behaviour as *x* becomes large positive and large negative.

7. Values of *x* near which y becomes large positive or large negative.

8. Regions where the curve is convex up or down.

9. Asymptotes of the curve

10. Find whether the curve is symmetric

15.2 Behaviour as *x* becomes very Large

Suppose we have a function *f* defined for all sufficiently larger numbers. Then we get substantial information concerning our function by investigating how it behaves as x becomes large.

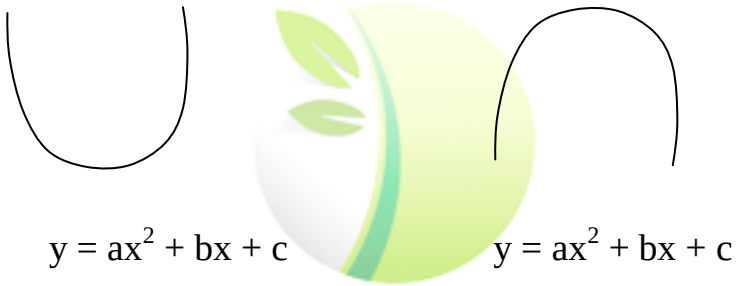
For example, sin *x* oscillates between -1 and +1 no matter how large x is.

However, polynomials do not oscillate. When $f(x) = x^2$ as x becomes large positive. So does x^2. Similarly with the function x^3, or x^4 (etc.). We consider this systematically.

Example 15.1 Consider a parabola,

$$y = ax^2 + bx + c, \text{ with } a \neq 0.$$

There are two essential cases, when $a > 0$ or $a < 0$. We have the parabola which looks like in the figure



$y = ax^2 + bx + c$ $\quad$ $y = ax^2 + bx + c$

$a > 0$ $\quad$ $a < 0$

We look some numerical examples.

Example 15.2 Sketch the graph of the curve

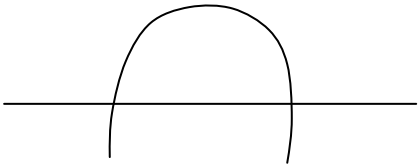
$$y = f(x) - 3x^2 + 5x - 1$$

We recognize this as a parabola.

$$f(x) = x^2(-3 + \frac{5}{x} - \frac{1}{x^2}),$$

when x is large positive or negative, then x^2 is large positive and the factor on

the right is close to -3. Hence f(x) is large negative. This means that the parabola has the shape as shown in figure.



We have $f'(x) = -6x + 5$. Thus $f'(x) = 0$ iff $x = \frac{5}{6}$. There is exactly one critical point. We have $f\left(\frac{5}{6}\right) = -3\left(\frac{5}{6}\right)^2 + \frac{25}{6} - 1 > 0$

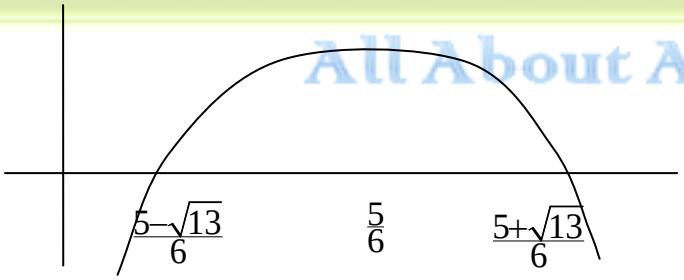
The critical point is a maximum, because we have already seen that the parabola bends down.

The curve crosses the x-axis exactly when

$$-3x^2 + 5x - 1 = 0$$

$$x = \frac{-5 \pm \sqrt{25-12}}{-6} = \frac{5 \pm \sqrt{13}}{6}$$

Hence the graph of the parabola looks as on the figure.



Bending down or convex upward

The same principle applies to sketching any parabola.

(i) Looking at what happens when *x* becomes large positive or negative tells us whether the parabola bends up or down.

(ii) A quadratic function

$$f(x) = ax^2 + bx + c \quad \text{with a} \neq 0$$

has only one critical point, when

$$f'(x) = 2ax + b = 0$$

So $x = \dfrac{-b}{2a}$

Knowing whether the parabola bends up or down tells us whether the critical point is maximum or minimum, and the value $x = \dfrac{-b}{2a}$ tells us exactly where this critical point lies.

(iii) The points where the parabola crosses the x-axis are determined by the quadratic formula.

Example 15.3. (Cubics) Consider a polynomial

f(x) = x^3 + 2x – 1, find *f(x)* when $x \to \pm\infty$. We have

We can write it in the form

$$x^3\left(1+\frac{2}{x^2}-\frac{1}{x^3}\right) \text{and, when} \, x \to +\infty \, \text{means} \;\; f(x) \to +\infty$$

Example 15.4. (a) Consider the quotient polynomials like

$$Q(x) = \frac{x^3 + 2x - 1}{2x^3 - x + 1}$$

Here if $x \to \pm\infty$, then $Q(x) \to \dfrac{1}{2}$.

Example 15.4(b) Consider the quotient $Q(x) = \dfrac{x^3 - 1}{x^2 + 5}$

Here $\lim_{x \to +\infty} Q(x) = +\infty$ *and* $\lim_{x \to -\infty} Q(x) = -\infty$

The meaning of the above limit is that there is no number which is the limit of Q(x) as $x \to +\infty$ or $x \to -\infty$.

We can now sketch the graphs of cubic polynomials symmetrically.

Example 15.5 Sketch the graph of $f(x) = x^3 - 2x + 1$

1. If $x \to +\infty$ then $f(x) \to +\infty$

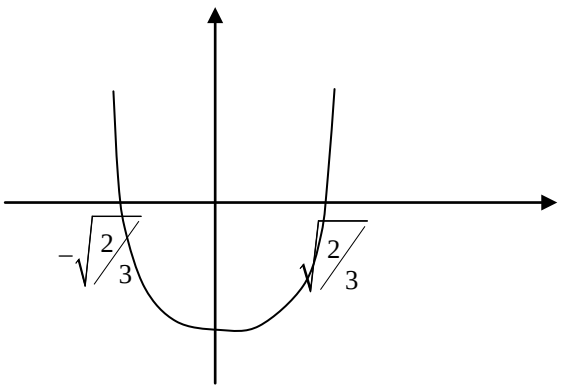
   If $x \to -\infty$ then $f(x) \to -\infty$

2. We have $f'(x) = 3x^2 - 2$

$$f'(x) = 0 \Leftrightarrow x = \pm\sqrt{2/3}$$

The critical points of f are $x = +\sqrt{2/3}$ and $x = -\sqrt{2/3}$.

3. Let $g(x) = f'(x) = 3x^2 - 2$. Then the graph of g is a parabola which is given as



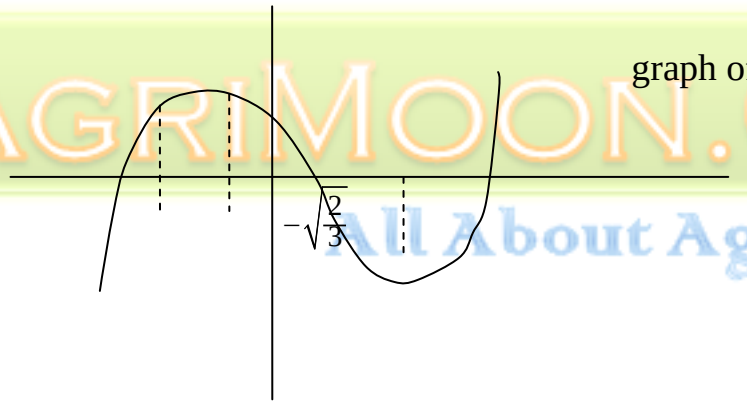
Graph of $g(x) = f'(x)$

Therefore, $f'(x) > 0 \Leftrightarrow x > \sqrt{2/3}$ *and* $x < -\sqrt{2/3}$, where g(x) > 0 and f is strictly increasing on the intervals $x \geq \sqrt{2/3}$ and $x \leq -\sqrt{2/3}$.

Similarly $f'(x) < 0 \Leftrightarrow -\sqrt{2/3} < x < \sqrt{2/3}$ where g(x) < 0, and f is strictly decreasing on this interval . Therefore $-\sqrt{2/3}$ is a local maximum for *f*, and $\sqrt{2/3}$ is a local maximum.

4. $f''(x) = 6x$, and $f''(x) > 0$ iff $x > 0$ and $f''(x) < 0$ iff for $x > 0$, therefore *f* is bending up (convex downward) for $x > 0$ and bending down (convex upward) for $x < 0$. There is an inflection point at $x = 0$.

Putting all this together, we find that the graph of f looks like this



Example 15.6 Sketch the graph of the curve.

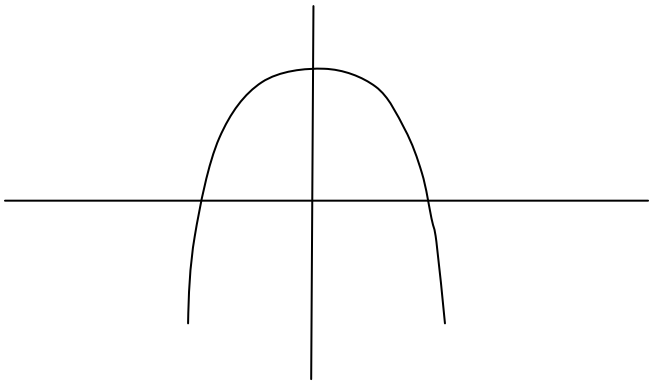
$y = -x^3 + 3x - 5$

1. When x = 0 , we have y = -5. With general polynomial for degree ≥ 3 there is in general no simple formula for those x such that f(x) = 0, so we do not give explicitly in the intersection of the graph with the *x* – axis.

2. The derivative is $f'(x) = -3x^2 + 3$

$$f'(x) = 0 \Leftrightarrow x = \pm 1$$

The graph of $f'(x)$ is given by



$f''(x) = -6x$, $f''(1) = -6$, $f''(-1) = 6$, $f''(x) > 0$ iff x < 0 and $f''(x) < 0$ iff x > 0. x = 0 is an inflection point x = 0.

f is strictly decreasing $\Leftrightarrow$ $f'(x) < 0$

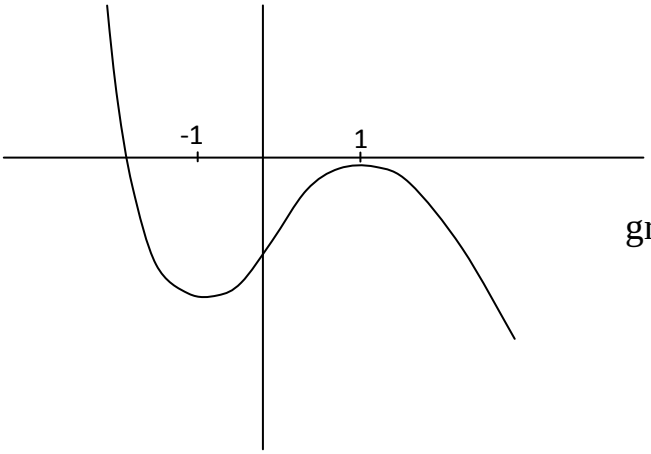
$\Leftrightarrow$ $x < -1$ *and* $x > 1$

f is strictly increasing $\Leftrightarrow$ $f'(x) > 0$

$\Leftrightarrow$ $-1 < x < 1$.

Therefore f has a local minimum at x = -1 and local maximum at x = 1.

Putting all this information together, we see that graph of f looks like this



graph of $f(x) = -x^3 + 3x + 5$

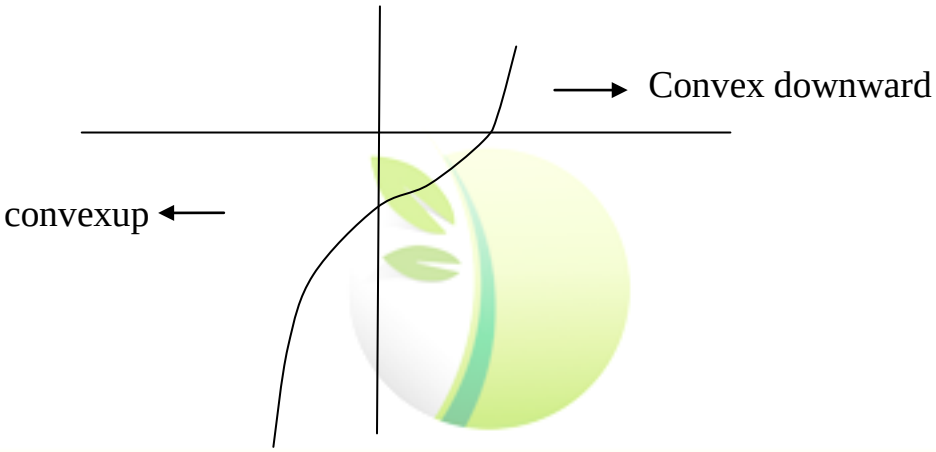
Example 15.7 Let f(x) = $4x^3$+2. Sketch the graph of f.

Solution:

Here we have $f'(x) = 12x^2 > 0 \ \forall \ x \neq 0$. There is only one critical point, when x = 0. Hence the function is strictly increasing for all x, and its graph looks like

$f''(x) = 24x > 0 \ \ for \ all \ \ x > 0$

$f''(x) < 0 \ \ for \ \ x < 0$



Example 15.8 Sketch the graph of $f(x) = 4x^3 + 4x$.

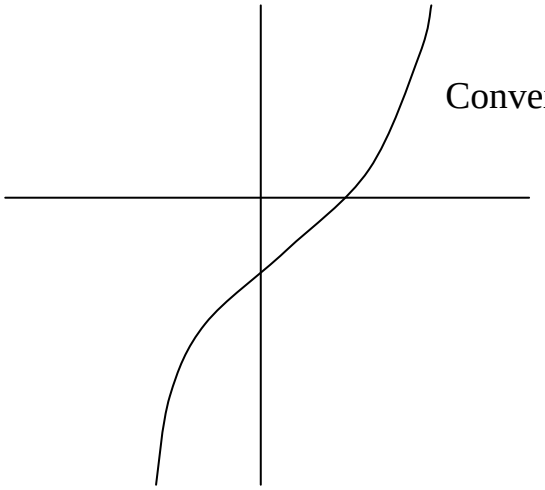
Solution:

$f'(x) = 3x^2 + 4 > 0 \ \forall x$

$f''(x) = 6x > 0 \ \ for \ \ x > 0$

$f''(x) < 0 \ \ for \ \ x < 0$

So the graph looks like



Convex downward

Convex upward

In both the above examples x = 0 is an inflection point.

15.3 Rational Functions

We shall now consider quotient of polynomials.

Example 15.9 Sketch the graph of the curve

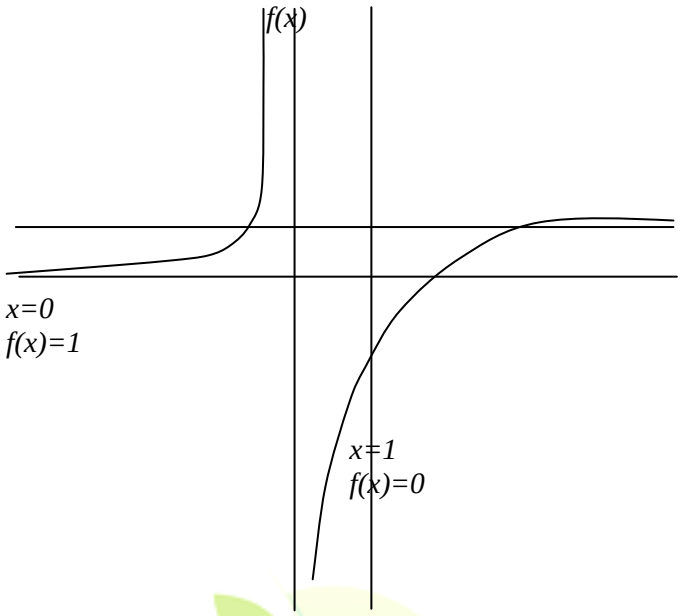
$$y = f(x) = \frac{x-1}{x+1}$$

1. When x = 0, we have f(x) = 1. When x = 1, f(x) = 0.

2. The derivative is $f'(x) = \frac{2}{(x+1)^2}$

It is never zero, so the function has no critical points.

3. The denominator is a square and hence is always positive, whenever it is defined, i.e., for $x \neq -1$. Thus $f'(x) > 0$ for $x \neq -1$. The function is not defined at x = -1 and hence derivative also is not defined at x = -1, i.e., f(x) is increasing in the region x < -1 and is increasing in the region x > -1



4. There is no region of decreasing.

5. Since the derivative is never zero, there is no relative maximum or minimum.

6. The second derivative is $f''(x) = \frac{-4}{(x+1)^3}$.

There is no inflection point since $f''(x) \neq 0$ for all x where the function is defined. If $x < -1$, $(x + 1)^3 < 0$, and $f''(x) > 0$, f(x) is bending up or convex downward. If $x > -1$, then $x+1 > 0 \Rightarrow (x+1)^3 > 0$. So $f''(x) < 0$ i.e., *f(x)* is bending down (convex upward).

7. As $x \to \infty$, $f(x) \to 1$ $\quad f(x) = \frac{x-1}{x+1} = \lim_{x\to\infty} \frac{x\left(1-\frac{1}{x}\right)}{x\left(1+\frac{1}{x}\right)} = 1$

when $x \to -\infty$, $f(x) \to 1$

8. As $x \to -1$, the denominator approaches 0 and the numerator approaches -2. If x approaches -1 from the right so x > -1, then the denominator is +ve and the numerator is negative. Hence the function $\frac{x-1}{x+1}$ is negative , and is large negative. Putting all these information we get the graph looks like the given figure.

EXERCISES

Sketch the following curves, indicating all the information stated in the examples etc.

1. $y = \frac{x^2+2}{x-3}$

2. $y = \frac{x-3}{x^2+1}$

3. $y = x^4 + 4x$

4. $y = x^8 + x$

5. $f(x) = x^4 + 3x^3 - x^2 + 5$

6. $y = \frac{x^2-1}{x}$

7. Show that a curve $y = ax^3 + bx^2 + cx + d$ with $a \neq 0$ has exactly one inflection point.

Keywords: Curve tracing, increasing, decreasing, convex up, convex down.

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Suggested Readings

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Lesson 16

Improper Integral

16.1 Introduction

Integral with infinite limits. Let a function f(x) be defined, positive and continuous for all values of x such that $a \le x < \infty$. Consider the integral

$$I(b) = \int_a^b f(x)\,dx$$

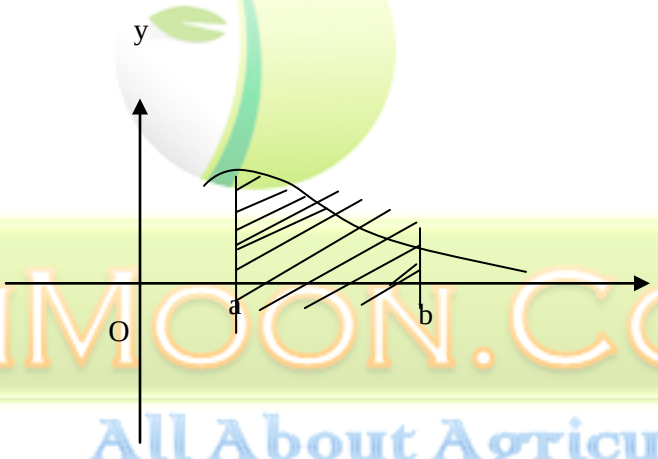

Fig. 1

This integral is meaningful for $b > a$. This integral varies with b and is continuous function of b. Let us consider the behavior of this integral when $b \to +\infty$ (Fig. 1). Definition 16.1 if there exists a finite limit

$$\lim_{b\to\infty} \int_a^b f(x)\,dx$$

Then this limit is called the improper integral of the function f(x) on the interval $[a,+\infty]$ and is denoted by the symbol

$$\int_a^{+\infty} f(x)\,dx$$

Thus, by definition, we have

$$\int_a^{+\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$

In this case it is said that the improper integral exists or converges. If $\int_a^b f(x)\,dx$ as $b \to +\infty$ does not have a finite limit, one say that $\int_a^{+\infty} f(x)\,dx$ does not exist or diverges.

If $f(x) \geq 0$, the geometrical meaning of the improper integral can be seen as if the integral $\int_a^b f(x)\,dx$ expresses the area of region bounded by the curve y = f(x), the x – axis and the ordinates x = a, x = b, it is natural to consider that the improper integral $\int_a^{+\infty} f(x)\,dx$ expresses the area of an unbounded (infinite) region lying between the curve $y = f(x)$, $x = a$ and x-axis.

We similarly define the improper integrals of other infinite intervals:

$$\int_{-\infty}^a f(x)\,dx = \lim_{\alpha\to-\infty}\int_\alpha^a f(x)\,dx$$

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{-\infty}^{c} f(x)\,dx$$

The latter equation should be understood as if each of the improper integrals on the right exists, then, by definition, the integral on the left also exists (converges).

Example 16.1: Evaluate the integral $\int_0^{+\infty} \frac{dx}{1+x^2}$

Solution:

By the definition of improper integral we find

$$\int_0^{+\infty} \frac{dx}{1+x^2} = \lim_{b\to\infty} \int_0^b \frac{dx}{1+x^2} = \lim_{b\to\infty} \tan^{-1} x \Big|_0^b = \frac{\pi}{2}$$

Note that this integral expresses the area of an infinite curvilinear trapezoid crosses x –axis as $x \to \infty$.

Example 16.2: Evaluate $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$

Solution:

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_0^{+\infty} \frac{dx}{1+x^2}$$

The 2nd integral is equal to $\frac{\pi}{2}$ (see example 1)

Compute the First Integral:

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{b\to-\infty} \int_{b}^{0} \frac{dx}{1+x^2} = \lim_{b\to-\infty} \tan^{-1} x\Big|_{b}^{0} = \lim_{b\to-\infty} \left(\tan^{-1} 0 - \tan^{-1} b\right) = \frac{\pi}{2}$$

Hence, $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

In many cases it is sufficient to determine whether the given integral converges or diverges, and to estimate its value. The following theorems, which we give without proof, may useful in this respect.

Theorem 16.1: Let f and g be continuous function on the interval $[a,\infty)$ with $o \le f(x) \le g(x) \ \forall\ a \le x < \infty$.

If $\int_{a}^{+\infty} g(x)\,dx$ converges then $\int_{a}^{+\infty} f(x)\,dx$ also converges, and

$$\int_{a}^{+\infty} f(x)\,dx \le \int_{a}^{+\infty} g(x)\,dx$$

Theorem 16.1: The integral of a discontinuous function:

The integral $\int_{a}^{c} f(x)\,dx$ of the function f(x) discontinuous at a point c is defined as follows:

$$\int_a^c f(x)\,dx = \lim_{\varepsilon \to 0+} \int_a^{c-\varepsilon} f(x)\,dx$$

If the limit on the right exists, the integral is called an improper convergent integral, otherwise it is divergent. If the function f(x) is discontinuous at x = a of the interval [a,c] then by definition ,

$$\int_a^c f(x)\,dx = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^c f(x)\,dx$$

If the function *f(x)* is discontinuous at some point $x = x_0$ inside the interval [a,c] , we put

$$\int_a^c f(x)\,dx = \int_a^{x_0} f(x)\,dx + \int_{x_0}^c f(x)\,dx$$

If both the improper integrals on the right hand side of the equation exist.

Example 16.3 Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x}}$

Solution: $\int_0^1 \frac{dx}{\sqrt{1-x}} = \lim_{\varepsilon \to 0} \int_0^{1-\varepsilon} \frac{dx}{\sqrt{1-x}}\,dx$

$$= \lim_{\varepsilon \to 0} \int_0^{1-\varepsilon} (1-x)^{-\frac{1}{2}}\,dx$$

$$= -\frac{(1-x)^{\frac{1}{2}}}{-\frac{1}{2}+1}\Bigg|_0^{1-\varepsilon}$$

$$= -2\sqrt{1-x}\Big|_0^{1-\varepsilon}$$

$$= \lim_{\varepsilon \to 0} -2\left(\sqrt{\varepsilon} - 1\right) = 2$$

Example 16.4: Evaluate the integral $\int_{-1}^{1} \frac{dx}{x^2}$.

Solution:

Since inside the interval of integration there exist a point x = 0, at which the integrand is not continuous, we express the integration as:

$$\int_{-1}^{1} \frac{dx}{x^2} = \lim_{\varepsilon \to 0} \int_{-1}^{0-\varepsilon} \frac{dx}{x^2} + \lim_{\varepsilon \to 0} \int_{0+\varepsilon}^{1} \frac{dx}{x^2}$$

$$= \lim_{\varepsilon \to 0} \int_{-1}^{-\varepsilon} \frac{dx}{x^2} + \lim_{\varepsilon \to 0} \int_{\varepsilon}^{1} \frac{dx}{x^2}$$

$$= \lim_{\varepsilon \to 0} -\frac{1}{x}\Big|_{-1}^{-\varepsilon} - \lim_{\varepsilon \to 0} \frac{1}{x}\Big|_{\varepsilon}^{1}$$

$$= -\lim_{\varepsilon \to 0}\left(-\frac{1}{\varepsilon} + 1\right) - \lim_{\varepsilon \to 0}\left(1 - \frac{1}{\varepsilon}\right)$$

But $-\lim_{\varepsilon \to 0}\left(-\frac{1}{\varepsilon} + 1\right) = \infty$ and $-\lim_{\varepsilon \to 0}\left(1 - \frac{1}{\varepsilon}\right) = \infty$ i.e., the integral diverges on [-1,0] as well as on [0,1] .

Hence the given integral diverges on the entire interval [-1, 1].

It should be noted that if we had evaluated the given integral without paying attention to the discontinuity of the integrand at point x = 0, the result would have been wrong as $\int_{-1}^{1} \frac{dx}{x^2} = \frac{-1}{x}\Big|_{-1}^{1} = -\left(\frac{1}{1} - \frac{1}{-1}\right) = -2$

For determining the convergence of improper integrals of discontinuous functions and for estimating their values, one can refer Lesson 17. These integrals are discussed in details in Lesson 17 also.

$\int_a^c \frac{dx}{(c-x)^p}$, *also* $\int_a^c \frac{dx}{(x-a)^p}$

It is easy to verify that $\int_a^c \frac{dx}{(c-x)^p}$ converges for p < 1 and diverges for p ≥ 1.

Same applies also to 2nd integral.

EXERCISES

Evaluate the following improper integrals:

1. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

2. $\int_0^\infty e^{-x} dx$

3. $\int_0^\infty \frac{dx}{a^2 + x^2}$

4. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

5. $\int_0^1 \ln x \, dx$

Ans.: 1. 1, 2. 1, 3. $\frac{\pi}{2a}\ (a>0)$, 4. $\frac{\pi}{2}$ & 5. 1

Keywords: Improper Integrals, Positive Function, Area Of the Region.

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Widder, D.V. (2002). Advance Calculus 2nd Edition, Publishers, PHI, India.

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Suggested Readings

Tom M. Apostol (2003). Calculus, Volume II Second Editions, Publishers, John Willey & Sons, Singapore.

Lesson 17

Tests for Convergence

17.1 Introduction

In this Lesson the convergence of Improper Integrals is studied.

Definition 16.1 if there exists a finite limit

$$\lim_{b\to\infty}\int_a^b f(x)\,dx$$

Then this limit is called the value of the improper integral of the function f(x) on the interval $[a,+\infty]$ and is denoted by the symbol

$$\int_a^{+\infty} f(x)\,dx$$

Thus, by definition, we have

$$\int_a^{+\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$

In this case it is said that the improper integral exists or converges. If $\int_a^b f(x)\,dx$ as $b\to+\infty$ does not have a finite limit, one say that $\int_a^{+\infty} f(x)\,dx$ does not exist or diverges.

If $f(x)\geq 0$, the geometrical meaning of the improper integral can be seen as if the integral $\int_a^b f(x)\,dx$ expresses the area of region bounded by the curve y =

f(x), the x – axis and the ordinates x = a, x = b, it is natural to consider that the improper integral $\int_a^{+\infty} f(x)\,dx$ expresses the area of an unbounded (infinite) region lying between the curve y = f(x), x = a and x-axis.

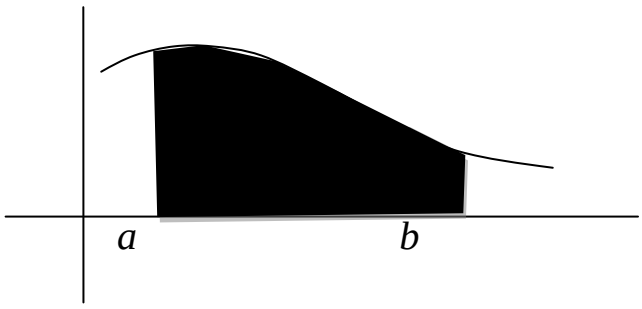
We similarly define the improper integrals of other infinite intervals:

$$\int_{-\infty}^{a} f(x)\,dx = \lim_{\alpha\to-\infty}\int_{\alpha}^{a} f(x)\,dx$$

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{-\infty}^{c} f(x)\,dx$$

The latter equation should be understood as if each of the improper integrals on the right exists, then, by definition, the integral on the left also exists (converges).

Example 16.1 Find out at which p the integral $\int_1^{+\infty} \frac{dx}{x^p}$ converges and at which it diverges.



Solution:

Since (when $p \neq 1$)

$$\int_1^b \frac{dx}{x^p} = \frac{1}{1-p} x^{1-p} \Big|_1^b = \frac{1}{1-p}\left(b^{1-p} - 1\right)$$

We have

$$\int_1^{+\infty} \frac{dx}{x^p} = \lim_{b \to +\infty} \frac{1}{1-p}\left(b^{1-p} - 1\right)$$

Consequently, with respect to like this integral we conclude that if p > 1, then $\int_1^{+\infty} \frac{dx}{x^p} = \frac{1}{p-1}$, and the integral converges.

If p < 1, then $\int_1^{+\infty} \frac{dx}{x^p} = \infty$ and integral diverges.

When p = 1, $\int_1^{+\infty} \frac{dx}{x^p} = \ln x \Big|_1^{+\infty} = \infty$, and the integral diverges.

Note: We call the p-integral $\int_1^{+\infty} \frac{dx}{x^p}$ converges for p > 1, and diverges for $p \le 1$. which is in the comparison test of improper integral used.

In many cases it is sufficient to determine whether the given integral converges or diverges, and to estimate its value. The following theorems, which we give without proof, may useful in this respect.

Theorem 17.1. Let f and g be continuous function on the interval $[a, \infty)$ with $o \le f(x) \le g(x) \ \forall \ a \le x < \infty$.

If $\int_a^{+\infty} g(x)\,dx$ converges then $\int_a^{+\infty} f(x)\,dx$ also converges, and $\int_a^{+\infty} f(x)\,dx \le \int_a^{+\infty} g(x)\,dx$

Example 17.2 Investigate the integral $\int_1^{+\infty} \frac{dx}{x^2(1+e^x)}$ for convergence.

Solution:

It will be noted that when $1 \le x$

$$\frac{1}{x^2(1+e^x)} < \frac{1}{x^2}$$

And $\int_1^{+\infty} \frac{dx}{x^2} = -\frac{1}{x}\Big|_1^{+\infty} = 1$

Consequently, $\int_1^{+\infty} \frac{dx}{x^2(1+e^x)}$ converges, and its value is less than 1. Hence $\int_1^{+\infty} \frac{dx}{x^2(1+e^x)}$ converges.

Theorem 17.2. If for all $x(x \ge a), 0 \le g(x) \le f(x)$ holds true and $\int_a^{+\infty} g(x)\,dx$ diverges, then the integral $\int_a^{+\infty} f(x)\,dx$ also diverges.

Example 17.3 Find out whether the following integral converges or diverges.

$$\int_1^{+\infty} \frac{x+1}{\sqrt{x^3}}\,dx$$

Solution:

We note that $\frac{x+1}{\sqrt{x^3}} > \frac{x}{\sqrt{x^3}} = \frac{1}{\sqrt{x}}$

But $\int_1^{+\infty} \frac{dx}{x^{\frac{1}{2}}} = \infty$ as $p = \frac{1}{2} < 2$. Hence the given integral is divergent.

In the above two theorems we considered improper integrals of nonnegative functions. For the case of a function f(x) which changes its sign over an infinite interval we have the following result.

Theorem17.3. If the integral $\int_a^{+\infty} |f(x)|dx$ converges, then the integral $\int_a^{+\infty} f(x)dx$ also converges.

In this case, the later integral is called an absolutely convergent integral.

Definition 17.1: An integral $\int_a^{+\infty} f(x)dx$ converges conditionally if and only if $\int_a^{+\infty} f(x)dx$ converges but $\int_a^{+\infty} |f(x)|dx$ is not convergent.

Example 17.3 Investigate the convergence of the integral $\int_1^{+\infty} \frac{\sin x}{x^3} dx$.

Solution:

Here, $\left|\frac{\sin x}{x^3}\right| \le \left|\frac{1}{x^3}\right|$. But $\int_1^{+\infty} \frac{1}{x^3} dx$ convergent as p = 3.

Therefore, the integral $\int_1^{\infty} \frac{\sin x}{x^3} dx$ also converges.

17.2 The Integral of a Discontinuous Function

A function f(x) is defined and continuous when $a \le x < c$, and either not defined or discontinuous when x = c. In this case, one cannot speak of the integral $\int_a^c f(x)\,dx$ as limit of integral sums, because f(x) is not continuous on [a, c] and for this reason the limit may not exist.

The integral $\int_a^c f(x)\,dx$ of the function f(x) discontinuous at a point c is defined as follows:

$$\int_a^c f(x)\,dx = \lim_{\varepsilon \to 0} \int_a^{c-\varepsilon} f(x)\,dx$$

If the limit on the right exists, the integral is called an improper convergent integral, otherwise it is divergent. If the function f(x) is discontinuous at x = a of the interval [a, c] then by definition,

$$\int_a^c f(x)\,dx = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^c f(x)\,dx$$

If the function f*(x)* is discontinuous at some point x = x_0 inside the integral [a, c] , we put

$$\int_a^c f(x)\,dx = \int_a^{x_0} f(x)\,dx + \int_{x_0}^c f(x)\,dx$$

If both the improper integrals on the right hand side of the equation exist.

Example 17.4 Test the convergence of the integral $\int_{-1}^{1} \frac{dx}{x^2}$.

Solution:

Since inside the interval of integration there exist a point x = 0, at which the integrand is not continuous, we express the integration as:

$$\int_{-1}^{1}\frac{dx}{x^2}=\lim_{\varepsilon\to 0}\int_{-1}^{0-\varepsilon}\frac{dx}{x^2}+\lim_{\varepsilon\to 0}\int_{0+\varepsilon}^{1}\frac{dx}{x^2}$$

$$=\lim_{\varepsilon\to 0}\int_{-1}^{-\varepsilon}\frac{dx}{x^2}+\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1}\frac{dx}{x^2}$$

$$=\lim_{\varepsilon\to 0}-\frac{1}{x}\Big|_{-1}^{-\varepsilon}-\lim_{\varepsilon\to 0}\frac{1}{x}\Big|_{\varepsilon}^{1}$$

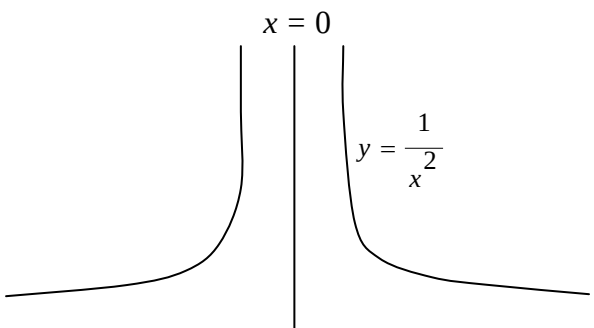
$$=-\lim_{\varepsilon\to 0}\left(-\frac{1}{\varepsilon}+1\right)-\lim_{\varepsilon\to 0}\left(1-\frac{1}{\varepsilon}\right)$$

But $-\lim_{\varepsilon\to 0}\left(-\frac{1}{\varepsilon}+1\right)=\infty$ and $-\lim_{\varepsilon\to 0}\left(1-\frac{1}{\varepsilon}\right)=\infty$ i.e., the integral diverges on [-1, 0] as well as on [0,1] .

Hence the given integral diverges on the entire interval [-1, 1].

It should be noted that if we had evaluated the given integral without paying attention to the discontinuity of the integrand at point x = 0, the result would have been wrong as $\int_{-1}^{1}\frac{dx}{x^2}=\frac{-1}{x}\Big|_{-1}^{1}=-\left(\frac{1}{1}-\frac{1}{-1}\right)=-2$

This is impossible (Fig. 3)



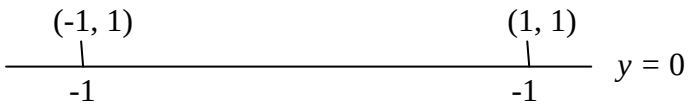


Fig. 3

Note: If the function f(x), defined on the interval [a, b], and has finite number of discontinuity points $a_1, a_2, \ldots, a_n$ within the interval,

Then $\int_a^b f(x)\,dx = \int_a^{a_1} f(x)\,dx + \int_{a_1}^{a_2} f(x)\,dx + \ldots.. + \int_{a_n}^b f(x)\,dx$

If each of the improper integrals on the right side of the equation converges then $\int_a^b f(x)\,dx$ is called convergent but if even one of these integrals diverges, then $\int_a^b f(x)\,dx$ too is called divergent.

For determining the convergence of improper integrals of discontinuous functions and for estimating their values, one can frequently make use of theorems similar to those used to estimate integrals within infinite limits.

Theorem 17.3. Let f(x) and g(x) be continuous functions in [a,c] except at x = c and at all points of this interval the inequalities $g(x) \geq f(x)$ hold and $\int_a^c g(x)\,dx$ converges, then $\int_a^c f(x)\,dx$ also converges.

Theorem 17.4. Let f(x) and g(x) be continuous functions on [a,c] except at x = c and at all points of this interval the inequalities $f(x) \geq g(x) \geq 0$ hold and $\int_a^c g(x)\,dx$ diverges, then $\int_a^c f(x)\,dx$ also diverges.

Theorem 17.5. Let f*(x)* be a continuous function on [a, c] except at $x = c$, and the improper integral $\int_a^c |f(x)|\,dx$ of the absolute value of this function converges, then the integral $\int_a^c f(x)\,dx$ of function of itself also converges. We frequently come across the improper integral of the following types.

$\int_a^c \frac{dx}{(c-x)^p}$, *also* $\int_a^c \frac{dx}{(x-a)^p}$

It is easy to verify that $\int_a^c \frac{dx}{(c-x)^p}$ converges for p < 1 and diverges for p ≥ 1.

Same applies also to 2nd.

Example17.5 Does the integral $\int_0^1 \frac{dx}{\sqrt{x+4x^3}}$ converge?

Solution:

The integrand is discontinuous at x = 0.

Now $\frac{1}{\sqrt{x+4x^3}} \leq \frac{1}{\sqrt{x}}$

The improper integral $\int_0^1 \frac{dx}{x^{\frac{1}{2}}}$ *as* $\frac{1}{2} < 1$ exists and hence $\int_0^1 \frac{dx}{\sqrt{x+4x^3}}$ also exists.

EXERCISES

Test the convergence of the following improper integrals:

1. $\int_0^\infty x \sin x \, dx$

2. $\int_1^\infty \frac{dx}{\sqrt{x}}$

3. $\int_{-\infty}^\infty \frac{dx}{x^2 + 2x + 2}$

4. $\int_0^1 \frac{dx}{x^{\frac{1}{3}}}$

5. $\int_0^2 \frac{dx}{x^3}$

6. Let b > 2. Find the area under the curve $y = e^{-2x}$ between 2 and b. Does this area approach a limit when $b \to \infty$. If so what limit?

7. Can an improper integral $\int_a^\infty f(x)\,dx$ ever be transformed onto a proper integral by a change of variable?

Ans.: 1. The integral diverges, 2. The integral diverges, 3. π, 4. $\frac{3}{2}$, 5. The integral diverges, 6. $-\frac{1}{2}e^{-2b} + \frac{1}{2}e^{-4}$, *yes* $\frac{1}{2}e^{-4}$ & 7. Yes, $f(x) = \frac{1}{x^2}$, $x = \frac{1}{t}$.

Keywords: Convergence, absolutely convergence, comparison test.

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Suggested Readings

Tom M. Apostol (2003). Calculus, Volume II Second Editions, Publishers, John Willey & Sons, Singapore.

Lesson 18

Rectification

18.1 Introduction

The method of finding the length of the arc of the curve of is called the rectification. Let $y = f(x)$ be a differentiable function defined on $[a,b]$ with $a < b$ and assume that its derivative is continuous. Our aim is to determine the length of the curve described by the graph. The main idea behind this is to approximate the curve by small line segments and add these up.

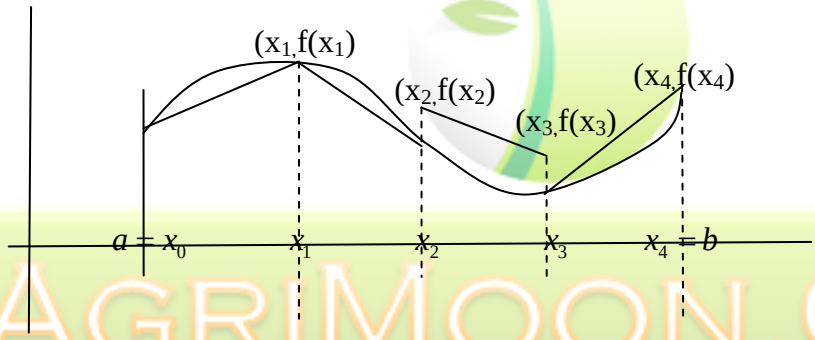


Fig .1

We consider a partition of the interval $[a,b]$. $a = x_0 \le x_1 \le x_2 \le x_3 ..... \le x_n = b$

In figure 1 take $n = 4$ for simplification.

For each x_i we have on the curve $(x_i, f(x_i))$. We draw the line segments between two successive points. The length of such a segments the length of the line between

$(x_i, f(x_i))$ and $(x_{i+1}, f(x_{i+1}))$ is equal to $\sqrt{(x_{i+1}-x_i)^2+(f(x_{i+1})-f(x_i))^2}$ ---------- (1)

$$(f(x_{i+1})-f(x_i))=(x_{i+1}-x_i)\,f'(c_i)$$

By mean value theorem, we conclude that

$$f(x_{i+1})-f(x_i)=(x_{i+1}-x_i)\,f'(c_i)\ \text{, where } c_i\in(x_i, x_{i+1})$$

Hence (1) becomes now

$$\sqrt{(x_{i+1}-x_i)^2+(x_{i+1}-x_i)^2\,f'(c_i)^2}$$

$$=\sqrt{(x_{i+1}-x_i)^2\left[1+f'(c_i)^2\right]}\ \text{---------- (18.2)}$$

Hence the form of the line segment is

$$\sum_{i=0}^{n-1}\sqrt{1+f'(c_i)^2}\,(x_{i+1}-x_i)\ \text{---------- (18.3)}$$

Now as $f'(x)$ is continuous function. So is $H(x)=\sqrt{1+f'(x)^2}$. So we can write eqn. (3) as $\sum_{i=0}^{n-1}H(c_i)(x_{i+1}-x_i)$

Since $H(x)$ is continuous on $[a,b]$, $H(c_i)$ satisfies the inequalities:

$$\min_{[x_i,x_{i+1}]}H\le H(c_i)\le\max_{[x_i,x_{i+1}]}H$$

i.e., $H(c_i)$ lies between the minimum and the maximum of on the interval $[x_i, x_{i+1}]$. Thus the sum we have written down lies between a lower sum and an upper sum for the function H . We call such sums as Riemann sums. This is true for every partition of the interval.

We know from basic integration theory that there is exactly one number lying between every upper sum and every lower sum, and that number is the definite interval. Therefore it is reasonable to define:

Length of our curve between a and b

$$= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{\left[1+ f'(x)^2\right]}\, dx \text{ ---------(18.4)}$$

Similarly for $x = \phi(y)$ and $\phi'(y)$ are continuous on $[a,b]$, then the length of our curve between a and b is

$$= \int_a^b \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy = \int_a^b \sqrt{1+\phi'(y)^2}\, dy$$

Example18.1 Find the length of the arc of $f(x) = x^{\frac{3}{2}}$ on [0, 4].

Solution:

As f, $f'(x)=\frac{3}{2}x^{\frac{1}{2}}$ are both continuous on [0, 4], the length of the arc or length

of curve L = $\int_0^4\sqrt{1+\left(\frac{3}{2}x^{\frac{1}{2}}\right)^2}\,dx=\int_0^4\sqrt{1+\frac{9}{4}x}\,dx$,

Let $1+\frac{9}{4}x=t$, when x=0, t=1,

x=4, t=10

$$\int_0^4\sqrt{1+\frac{9}{4}x}\,dx=\frac{4}{9}\int_1^{10}t^{\frac{1}{2}}dt=\frac{4}{9}\times\frac{2}{3}\times t^{\frac{3}{2}}\Big|_1^{10}=\frac{8}{27}\left[10^{\frac{3}{2}}-1\right]$$

Example 18.2 Find the length of the curve $y=x^2$ between x =0 and x =1.

Solution:

From the definition above, we see that the integral is

$$\int_0^1\sqrt{1+(2x)^2}\,dx=\int_0^1\sqrt{1+4x^2}\,dx \text{ set u =2x, du = 2dx}$$

When x=0, u=0, x=1, u=2

Hence $\int_0^1\sqrt{1+4x^2}\,dx=\frac{1}{2}\int_0^2\sqrt{1+u^2}\,du$ ---------- (18.5),

We can find the integral $\int_0^b\sqrt{1+x^2}\,dx$ for b > 0, as

$$\frac{1}{4}\left[\frac{1}{2}\left(b+\sqrt{b^2+1}\right)^2+2\ln\left(b+\sqrt{b^2+1}\right)-\frac{1}{2}\left(b+\sqrt{b^2+1}\right)^{-2}\right]$$

$$\text{So } \int_0^2 \sqrt{1+u^2}\,du = \frac{1}{4}\left[\frac{1}{2}\left(2+\sqrt{5}\right)^2 + 2\ln\left(2+\sqrt{5}\right) - \frac{1}{2}\left(2+\sqrt{5}\right)^{-2}\right]$$

Hence (18.5) becomes: $\frac{1}{8}\left[\frac{1}{2}\left(2+\sqrt{5}\right)^2 + 2\ln\left(2+\sqrt{5}\right) - \frac{1}{2}\left(2+\sqrt{5}\right)^{-2}\right]$

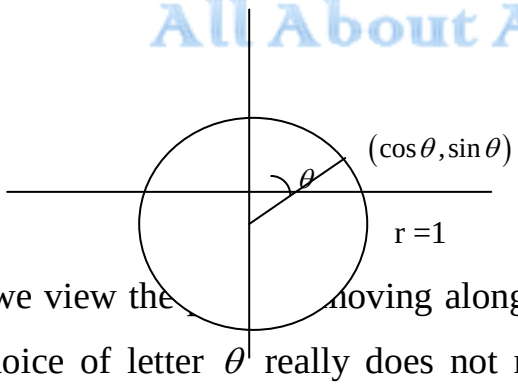
18.2 Length of Parameterized Curve

There is one other way in which we can describe a curve. Suppose that we look at a point which moves in the plane. Its coordinates can be given as a function of time t. Thus, we get two functions of t, say

$$x = f(t),\ \ y = g(t),$$

We may view these as describing a point moving along a curve. The functions *f* and g give the coordinates of the point as function of t.

Example 18.3 Let, $x = r\cos\theta$, $y = r\sin\theta$. Then $(x, y) = (r\cos\theta, r\sin\theta)$ is a point on the circle.



As θ increases, we view the point moving along the circle in anticlockwise direction. The choice of letter θ really does not matter and we could use t instead. In particular, the angle θ is itself express as a function of time. For example, if a bug moves around the circle with uniform (constant) angular speed, then we can write $\theta = \omega t$, where ω is constant.

Then $x = \cos(\omega t)$, $y = \sin(\omega t)$.

When (x, y) is described by two function of t as above, we say that we have a parameterization of the curve in terms of parameter t.

This describes the motion of a bug around the circle with angular speed ω. Note that the parametric representation of a curve is not unique. For example $x = r\sin\theta$, $y = r\cos\theta$ also represents a point on the circle.

We shall now determine the length of a curve given by a parameterization. Suppose that our curve is given by

$$x = f(t), y = g(t), \text{ with } a \le t \le b$$

and assume that both f, g have continuous derivatives. With eqn (18.4) it is very reasonable to define the length of our curve (in parametric form) to be

$$l_a^b = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\, dt\,.$$

Observe that when a curve is given in usual form y = f(x) we can let

$$t = x = g(t) \quad \text{and } y = f(t).$$

This shows how to view the usual form as a special case of the parametric form. In that case $g'(t) = 1$ and the formula for the length in parametric form is seen to be the same as the formula we obtained before for a curve $y = f(x)$ .It is also convenient to put the formula in the other standard notation for the derivative. We have

$$\frac{dx}{dt} = f'(t) \text{ and } \frac{dy}{dt} = y'(t)$$

Hence the length of the curve can be written in the form

$$l_a^b = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Without loss of generality let

s(t) = length of the curve as function of t.

Thus we may write

$$s(t) = \int_a^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

This gives

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{f'(t)^2 + g'(t)^2}$$

Sometimes one writes symbolically

$$(ds)^2 = (dx)^2 + (dy)^2$$

To suggest the Pythagoras theorem i.e.,

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Example 18.4 Find the length of the curve x = cos t , y = sin t between $t = 0, t = \pi$

Solution:

The length is the interval

$$\int_0^{\pi} \sqrt{(\sin t)^2 + (\cos t)^2}\, dt$$
$$= \pi$$

If we integrate between 0 and 2π we would get 2π. This is the length of the circle of radius 1.

Example 18.5 Find the length of the curve $x = e^t \cos t$, $y = e^t \sin t$ between t =1 and t = 2.

Solution:

$$l_1^2 = \int_1^2 \sqrt{\left[\left(e^t \cos t\right)'\right]^2 + \left[\left(e^t \sin t\right)'\right]^2}\, dt$$

$$= \int_1^2 \sqrt{(-e^t \sin t + e^t \cos t)^2 + (e^t \cos t + e^t \sin t)^2}\, dt$$

$$= \int_1^2 \sqrt{(e^{2t} \sin^2 t + e^{2t} \cos^2 t + e^{2t} \cos^2 t + e^{2t} \sin^2 t)}\, dt$$

$$= \sqrt{2}\int_1^2 e^t dt = \sqrt{2}\left[e^2 - e\right]$$

Example 18.6 Find the length of the curve

$x = \cos^3\theta \quad , \quad y = \sin^3\theta \quad \text{for} \quad 0 \le \theta \le \pi/2$

Solution:

We have $\dfrac{dx}{d\theta} = 3\cos^2\theta(-\sin\theta)$

$$\frac{dy}{d\theta} = 3\sin^2\theta\cos\theta$$

Hence,

$$l_0^{\frac{\pi}{2}} = \int_0^{\frac{\pi}{2}} \sqrt{9\cos^4\theta + 9\sin^4\theta\cos^2\theta}\ d\theta$$

$$= 3\int_0^{\frac{\pi}{2}} \sqrt{\cos^2\theta\sin^2\theta}\ d\theta$$

$$= 3\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\ d\theta \quad \text{as } \sin\theta, \cos\theta > 0 \quad \text{for } 0 \le \theta \le \pi/2$$

Hence

$$l_0^{\frac{\pi}{2}} = 3\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\ d\theta = \frac{3}{2}\int_0^{\frac{\pi}{2}} \sin 2\theta\ d\theta = -\frac{3}{4}\cos 2\theta\Big|_0^{\frac{\pi}{2}} = \frac{3}{2}$$

EXERCISES

Find the length of the following curves:

1. $y = \ln x$, $\frac{1}{2} \le x \le 2$,

2. $y = 4 - x^2$, $-2 \le x \le 2$,

3. $y = \frac{1}{2}(e^x + e^{-x})$ between $x = 1$ and $x = -1$

4. $y = \ln \cos x$, $0 \le x \le \frac{\pi}{3}$,

5. Find the length of the circle of radius r .

6. Find the length of the curve $x = \cos^3 t$, $y = \sin^3 t$ between $t = 0$ and $t = \pi$

7. Find the length of the curve $x = 3t$, $y = 4t - 1$, $0 \le t \le 1$.

8. Find the length of the curve $x = 1 - \cos t$, $y = t - \sin t$, $0 \le t \le 2\pi$.

9. Using exercise (9), find the length of the curve $r = \sin^2 \frac{\theta}{2}$ from 0 *to* π .

Ans.: 1. $\frac{\sqrt{5}}{2} + \ln\left(\frac{4+2\sqrt{5}}{1+\sqrt{5}}\right)$, 2. $2\sqrt{17} + \ln\left(\frac{\sqrt{17}+4}{\sqrt{17}-4}\right)^{\frac{1}{4}}$, 3. $e - \frac{1}{e}$, 4. $\ln\left(2+\sqrt{3}\right)$,

5. $2\pi r$, 6. 3, 7. 5, 8. 8 & 9. 2

Keywords: Rectification, length of curve, parametric form,

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Suggested Readings

Tom M. Apostol (2003). Calculus, Volume II Second Editions, Publishers, John Willey & Sons, Singapore.

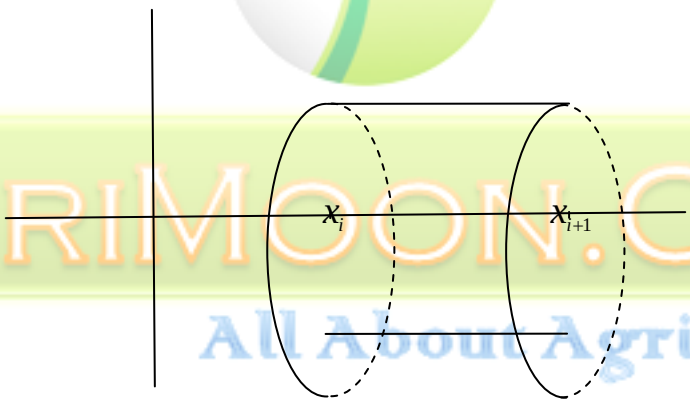
Lesson 19

Volume and Surface of Revolution

19.1 Introduction

Volume of Revolution: We start our applications with volumes of revolutions. Our aim is to find the lengths, areas and volumes of the standard geometric figures.

Let $y = f(x)$ be continuous function of x on the interval with $[a,b]$ with $(a < b)$. Assume that $f(x) \geq 0 \ \forall \ x \in [a,b]$. If we revolve $y = f(x)$ around axis, we obtain a solid, whose volume we want to compute.



Take a partition of $[a,b]$ say $a = x_0 \leq x_1 \leq x_2 \leq x_3 \leq .....x_n \leq = b$

Let c_i be a minimum of f on the interval $[x_i, x_{i+1}]$ and d_i be the maximum of f in that interval. Then the solid of revolutions is that small interval lies between a small cylinder and a big cylinder. The width of these cylinders is $x_{i+1} - x_i$ and the radius is $f(c_i)$ for the small cylinders and $f(d_i)$ for the big cylinder. Hence the volume of revolutions, denoted by V satisfies the inequalities

$$\sum_{i=0}^{n-1} \pi f(c_i)^2 (x_{i+1} - x_i) \leq V \leq \sum_{i=0}^{n-1} \pi f(d_i)^2 (x_{i+1} - x_i)$$

It is therefore reasonable to define this volume to be $V = \int_a^b \pi f(x)^2 \, dx$

If we revolve the curve around $x = \phi(y)$ around $y-axis$ and $\phi(y) \geq 0 \ \forall \ y \in [c,d]$, we define the volume to be $V = \int_c^d \pi f(y)^2 \, dy$

If the curve be expressed by $x = f(t),\ y = \phi(t)$ $V = \pi \int_a^b y^2 \, dx = \pi \int_{t_1}^{t_2} (\phi(t))^2 \, f'(t) \, dt$ where t_1, t_2 are values of t that corresponds to $x = a$ and $x = b$ respectively.

Example 19.1: Compute the volume of the sphere of radius 1.

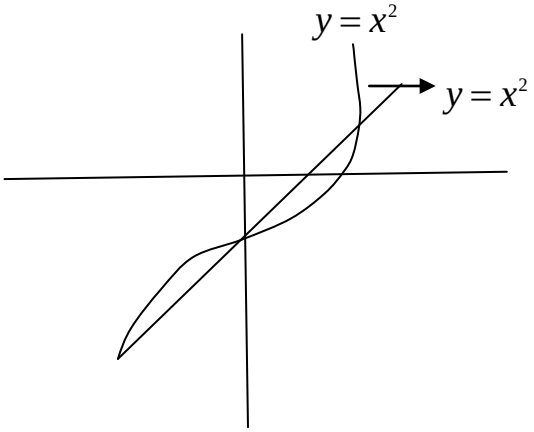
Solution:

We take the function $y = \sqrt{1-x^2}$ between 0 and 1. If we rotate this curve around $x-axis$, we shall get half the sphere. Its volume is therefore

$$\int_0^1 \pi(1-x^2)dx = \pi\left(x - \frac{x^3}{3}\right)\Big|_0^1 = \frac{2}{3}\pi$$

So the volume of full sphere is $2 \times \frac{2}{3}\pi = \frac{4}{3}\pi$

Example 19.2: Find the volume obtained by rotating the region between $y = x^3$ and $y = x$ in the first quadrant around the $x-axis$.



The graph of the region is given on the figure.

As $x^3 = x \Rightarrow x(x^2 - x) = 0 \Rightarrow x = 0, x = \pm 1$, for first quadrant we take $0 \le x \le 1$. The required V volume is equal to the difference of the volume obtained by rotating $y = x$ and $y = x^2$.

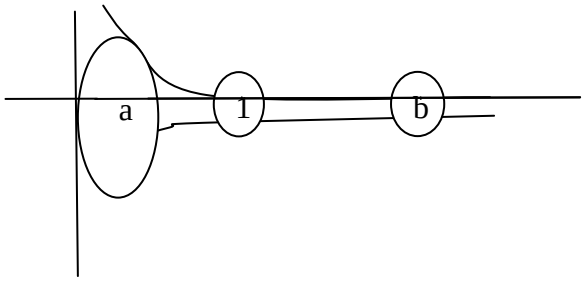
Let $f(x) = x, g(x) = x^3$. Then

$$V = \pi \int_0^1 f(x)^2 dx - \pi \int_0^1 g(x)^2 dx$$

$$= \pi \int_0^1 x^2 dx - \pi \int_0^1 x^6 dx$$

$$= \frac{\pi}{3} - \frac{\pi}{7}$$

Example 19.3: (Volume of Chimneys) . Consider the function $f(x) = \frac{1}{\sqrt{x}}$.



Let $0<a<1$. The volume of revolution of the curve $y=\frac{1}{\sqrt{x}}$ between $x=a$ and $x=1$ is given by $\int_a^1 \pi \frac{dx}{x} = \pi \ln x\Big|_a^1 = -\pi \ln a$,

As $a \to 0$, $\ln a$ becomes very large negative, so that $-\ln a$ becomes very large positive, and the volume becomes arbitrary large. The above figure illustrates the chimney.

In this computation, we determined the volume of a chimney near the $y-axis$. We can also fixed the volume of the chimney going off to the right, say between 1 and a number $b>1$. Suppose the chimney is defined by $y=\frac{1}{\sqrt{x}}$. The volume of revolution between 1 and b is given by the integral $\int_1^b \pi\left(\frac{1}{x}\right)dx = \int_0^b \pi \frac{dx}{x} = \pi \ln b$,as $b \to \infty$ we see that this volume becomes arbitrary large (divergent integral)

But we are interested to find finite volume for the infinite chimney.

Example 19.4: Compute the volume of revolution of the curve $y=\frac{1}{x^4}$ between a and 1. Find the limit as $a \to 0$

Solution:

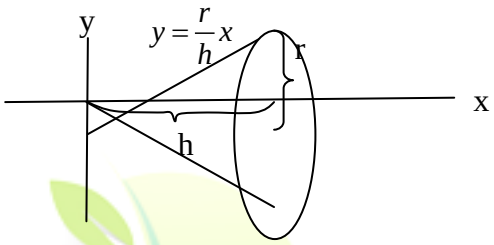
The volume of revolution of the curve $y=\frac{1}{x^4}$ between $x=a$ and $x=1$

is given by the integral $\int_a^1 \pi \frac{1}{x^{\frac{1}{2}}} dx = \pi \int_a^1 x^{-\frac{1}{2}} dx = \pi \times 2 x^{\frac{1}{2}}\Big|_a^1 = 2\pi\left[1-\sqrt{a}\right]$

When $a \to 0$ limit becomes 2π

Example 19.5 Find the volume of a cone whose base has a radius r , and a height h , by rotating a straight line passing through the origin around the $x-axis$

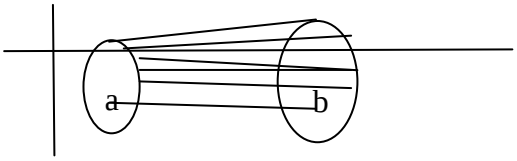
Solution:



The equation of the straight line is $y=\frac{r}{h}x$. Slant height is $y=\frac{1}{x^2}$. Hence the volume of the cone is $\int_0^h \pi\left(\frac{r}{h}x\right)^2 dx=\pi\frac{r^2}{h^2}\int_0^h x^2dx=\frac{\pi r^2}{h^2}\times\frac{h^3}{3}=\frac{1}{3}\pi r^2h$

19.2 Surface of Revolution

Let $y=f(x)$ be a positive continuously differentiable function on an interval [a,b]. We wish to find a formula for the area of the surface of revolution of the graph of f around the $x-axis$, as given in the figure

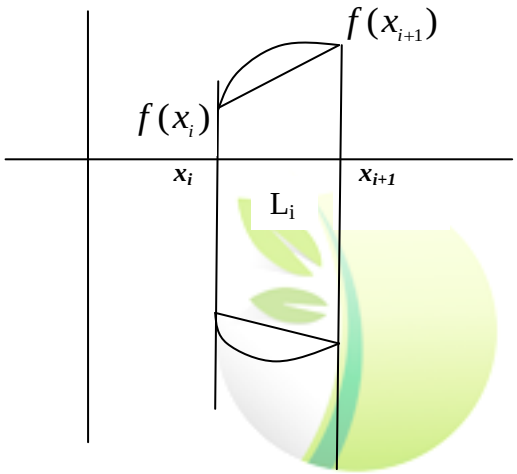


We shall see that the surface area is given by the integral

$$S = \int_a^b 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

The idea again is to approximate the curve by line segments. We use a partition

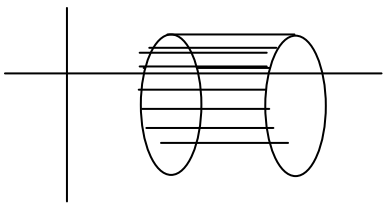
$a = x_0 \le x_1 \le x_2 \le x_3 ..... \le x_n = b$



On the small interval $[x_i, x_{i+1}]$ the curve is approximated by the line segment joining the points $(x_i, f(x_i))$ *and* $(x_{i+1}, f(x_{i+1}))$. Let L_i be the length of the segment.

Then $L_i = \sqrt{(x_{i+1}-x_i)^2 + (f(x_{i+1})^2 - f(x_i))^2}$

The length of a circle of radius y is $2\pi y$. If we rotate the line segment about the then the $x-axis$ area of the surface of rotation will be between $2\pi f(t_i)L_i$ and $2\pi f(s_i)L_i$ where $f(t_i)$ and $f(s_i)$ are the minimum and maximum of f, respectively on the interval $[x_i, x_{i+1}]$. This is illustrated on Fig 1.



On the other hand, by the mean value theorem we can write

$$f(x_{i+1}) - f(x_i) = f'(c_i)(x_{i+1} - x_i)\ ,\ c_i \in (x_i, x_{i+1})$$

Hence $L_i = \sqrt{(x_{i+1} - x_i)^2 + f(c_i)^2 (x_{i+1} - x_i)^2}$

$$= \sqrt{1 + f'(c_i)^2}\,(x_{i+1} - x_i)$$

Therefore the expression $2\pi f(c_i)\sqrt{1 + f'(c_i)^2}\,(x_{i+1} - x_i)$

is an approximation of the surface of revolution of the curve over the small interval $[x_i, x_{i+1}]$.

Now take the sum $\sum_{i=0}^{n-1} 2\pi f(c_i)\sqrt{1 + f'(c_i)^2}\,(x_{i+1} - x_i)$

This is a Riemann sum, between the upper and lower sums for the integral

$$S = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\,dx$$

Thus it is reasonable that the surface area should be defined by this integral, as was to be shown.

19.2.1 Area of revolution for parametric curves given in parametric form.

Suppose that

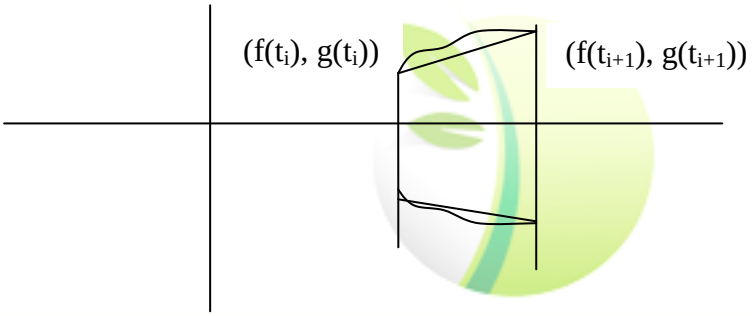
$x = f(t),\ y = g(t),\ a \le t \le b$

We take a partition $a = t_0 \le t_1 \le t_2 \le t_3 ..... \le t_n = b$

Then the length of L_i between $(f(t_i), g(t_i))$ and $(f(t_{i+1}), g(t_{i+1}))$ is given by

$$L_i = \sqrt{(f(t_{i+1}) - f(t_i))^2 + (g(t_{i+1}) - g(t_i))^2}$$
$$= \sqrt{f'(c_i)^2 + g'(d_i)^2}\,(t_{i+1} - t_i)$$

where c_i, d_i are numbers between t_i and t_{i+1}



Hence $2\pi g(c_i)\sqrt{f'(c_i)^2 + g'(d_i)^2}\,(t_{i+1} - t_i)$ is an approximation for the surface of revolution of the curve in the small interval $[t_i, t_{i+1}]$. Consequently, it is

reasonable that the surface of revolution is given by the integral

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

when $t = x$, this coincides with the formula found previously. It is also useful to write this formula symbolically $S = \int 2\pi y ds$

where symbolically, we had used

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

When using this symbolic notation, we don not put limits of integration. Only when we use explicit parameter over an interval $a \le t \le b$ we explicitly write the surface area as

$$S = \int_a^b 2\pi y \frac{ds}{dt} dt$$

Example 19.6 We wish to find the area of a sphere for radius $r > 0$.

Solution: we can view the sphere as the area of revolution of a circle for radius r, and to express the circle in parametric form,

$$x = r\cos\theta, y = r\sin\theta, 0 \le \theta \le \pi$$

Then the formula gives

$$S = \int_0^{\pi} 2\pi r \sin\theta \sqrt{r^2 \sin\theta + r^2 \cos\theta}\, d\theta$$

$$= \int_0^{\pi} 2\pi r^2 \sin\theta\, d\theta$$

$$= 2\pi r^2 (-\cos\theta)\Big|_0^{\pi}$$

$$= 4\pi r^2$$

Exercises

1. Find the volume of sphere of radius r.

Find the volumes of revolution of the following:

2. $y = \frac{1}{\cos x}$ between $x = 0$ and $x = \frac{\pi}{4}$

3. $y = \sin x$ between $x = 0$ and $x = \frac{\pi}{4}$

4. The region between $y = x^2$ and $y = 5x$

5. $y = xe^{\frac{x}{2}}$ between $x = 0$ and $x = 1$

6. Compute the volume of revolution of the curve $y = \frac{1}{x^2}$ between $x = 2$ and $x = b$ for any b>2. Does this volume approach a limit as $b \to \infty$? If yes, what limit ?

Ans.: 1. $\frac{4}{3}\pi r^3$, 2. π, 3. $\frac{\pi^2}{8} - \frac{\pi}{4}$, 4. $\frac{2.5^4\pi}{3}$, 5. $\pi(e-2)$ & 6. $\frac{\pi}{24} - \frac{\pi}{3b^3}$, yes: $\frac{\pi}{24}$

Keywords: Lengths, area, volume, surface revolution, volume of chimneys

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Suggested Readings

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Lesson 20

Double Integration

20.1 Introduction

In applications of calculus we have seen with integrals of functions of a single variable. The integral of a function y = f(x) over an interval [a, b] is the limit of approximating sums

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x_k \text{ ---------- (20.1)}$$

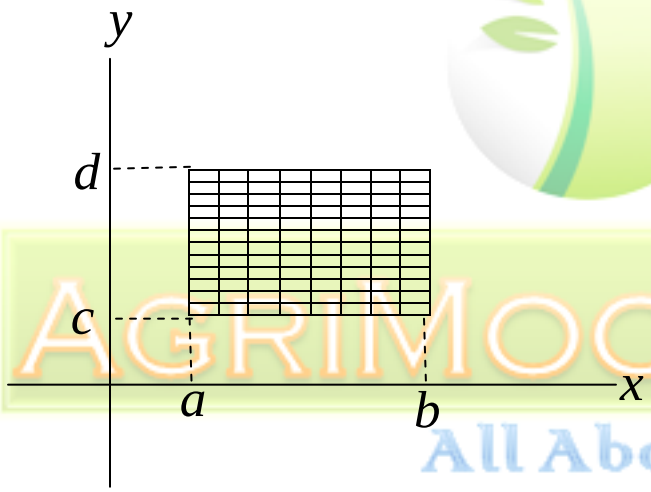
Where $a = x_0 \le x_1 \le x_2 \le \ldots\ldots \le x_n = b,\ \Delta x_k = x_{k+1} - x_k$ and c_k is the any point from the interval $[x_k, x_{k+1}]$. The limit in (20.1) is taken as the length of the longest subinterval approaches zero. The limit is guaranteed to exist if f is continuous and also exists when f is bounded and has only finitely many points of discontinuity in [a, b] . There is no loss in assuming the intervals $[x_k, x_{k+1}]$ to have common length $\Delta x = \frac{b-a}{n}$, and limit may thus obtain by letting $\Delta x = 0$ as $n \to \infty$.

If f(x) > 0, then $\int_a^b f(x)dx$ from x = a and x = b, but in general the integral has many other important interpretations (distance, volume, arc length, surface area, moment of inertia, mass, hydrostatic pressure, work) depending on the nature and interpretation of f.

In this Lesson we shall see that integrals of functions of two or more variables which are called multiple integrals and defined I much the same way as integrals of functions of single variable.

Double Integrals: Here we define the integral of a function f(x, y) of two variables over a rectangular region in xy-plane. We then show how such an integral is evaluated and generalize the definition to include bounded regions of a more general nature.

Double Integrals over Rectangles:



Suppose that f(x, y) is defined on a rectangular region R defined by

$R : a \le x \le b,\ c \le y \le d$

(see the figure 1.)

We imagine R to be covered by a network of lines parallel to x-axis and y-axis, as shown in Fig 1. These lines divide R into small pieces of area

$$\Delta A = \Delta x \Delta y$$

We number these in some order

$$\Delta A_1, \Delta A_2, ..., \Delta A_n,$$

Choose a point (x_k, y_k) in each piece of ΔA_k and from the sum

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k \text{ ---------- (20.2)}$$

If f is continuous throughout R, then we define mesh width to make both Δx and Δy go to zero the sums in (2) approach a limit called the double integral of f over R that is denoted by $\iint_R f(x,y)dA \text{ or } \iint_R f(x,y)dx\,dy$

Thus $\iint_R f(x,y)dA = \lim_{\Delta A \to 0} \sum_{1}^{n} f(x_k, y_k)\Delta A_k$ ---------- (20.3)

As with functions of a single variable, the sums approach this limit no matter how the interval [a, b] and [c, d] that determine R are subdivide, along as the lengths of the subdivisions both go to zero. The limit (20.3) is independent of the order in which the area ΔA_k are numbered, and independent of the choice of (x_k, y_k) within each ΔA_k. The continuity of f sufficient condition or the existence of the double integral, but not a necessary one, and limit question exists for many discontinuous functions also.

20.1.1 Properties of Double Integral

Like "single" integrals, we have the following properties for double integrals of continuous functions which are useful in computations and applications.

(i) $\iint_R k\, f(x,y)dA = k\iint_R f(x,y)dA$ (any number k)

(ii) $\iint_R [\, f(x,y) + g(x,y)]dA = \iint_R f(x,y)dA + \iint_R g(x,y)dA$

(iii) $\iint_R [\, f(x,y) - g(x,y)]dA = \iint_R f(x,y)dA - \iint_R g(x,y)dA$

(iv) $\iint_R f(x,y)dA \geq 0 \; if \;\; f(x,y) \geq 0 \; on \; R$

(v) $\iint_R f(x,y)dA \geq \iint_R g(x,y)dA \; if \;\; f(x,y) \geq g(x,y) \, on \; R$

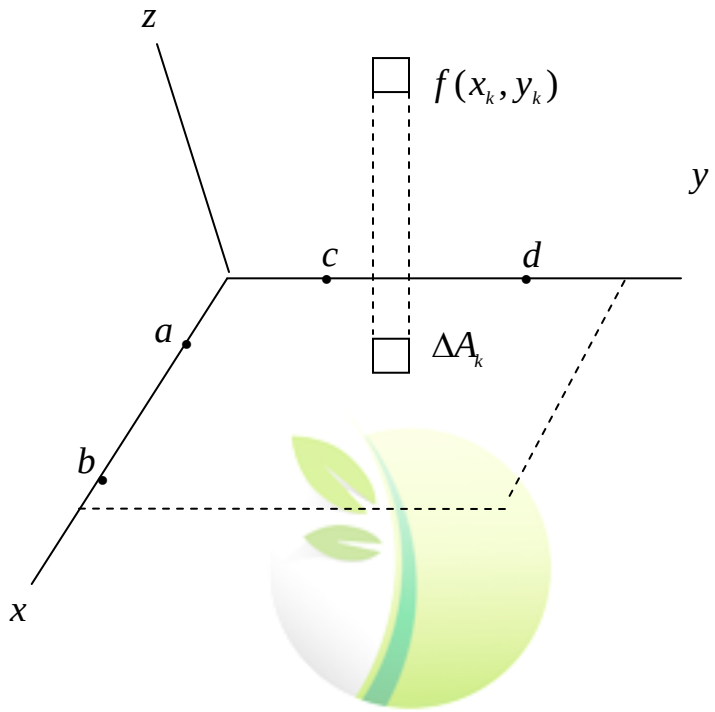
(vi) If $R = R_1 \cup R_2 , R_1 \cap R_2$, R is the union of two non-overlapping rectangles R_1 and R_2, we have

$$\iint_{R_1 \cup R_2} f(x,y)dA = \iint_{R_1} f(x,y)dA + \iint_{R_2} f(x,y)dA$$

Volume: When f(x, y) > 0, we may interpret $\iint_R f(x,y)dA$ as the volume of the solid enclosed by R, the planes x = a, x = b, y = c, y = d , and the surface z = f(x, y) see fig 2.

Each term $f(x_k, y_k)\Delta A_k$ in the sum

$S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$ is the volume of a vertical rectangular prism y that



approximate the volume of the portion of the solid that stands above the box - ΔA_k . The sum S_n thus approximates what we call the total volume of the solid, and we define this volume to be

Volume = lim S_n = $\iint_R f(x, y)dA$

20.1.2 Fubbin's theorem for calculating double integrals:

Theorem 20.1. (Fubbin's theorem (1st form))

If f(x, y) is continuous on the rectangular region $R: a \le x \le b,\ c \le y \le d$, then

$$\iint_R f(x, y)dA = \int_c^d \int_a^b f(x, y)dxdy = \int_a^b \int_c^d f(x, y)dydx$$

Fubbin's theorem shows that double integrals over rectangles can be calculated as iterated integrals. This means that we can evaluate a double integral by integrating one variable at a time, using the integration techniques we already know for function of a single variable.

Fubin's theorem also says that we may calculate the double integral by integrating in either order (a genuine convenience). In particular, when we calculate a volume by slicing, we may use either planes perpendicular to the x-axis or planes perpendicular to y-axis. We get same answer either way.

Even more important is the fact that Fubin's theorem holds for any continuous function f(x, y). In particular it may have negative values as well as positive values on R, and the integrals we calculate with Fubin's theorem may represent other things besides volumes.

Example 20.1: Suppose we wish to calculate the volume under the plane z = 4-x-y over the region $R: 0 \le x \le 2,\ 0 \le y \le 1$ in the xy – plane.

Solution: The volume under the plane is given by $\iint_R (4 - x - y)dA$.

Next we have to calculate the double integral.

Now we will complete the stated example.

$$\iint_R f(x, y)dA = \int_0^1 \int_0^2 (4 - x - y)dx\,dy$$

$$= \int_0^1 \left(4x - \frac{x^2}{2} - xy \right) \Big|_0^2 dy$$

$$= \int_0^1 (8 - 2 - 2y)\, dy$$

$$= \int_0^1 (6 - 2y)\, dy$$

$$= 6y - y^2 \Big|_0^1 = 5$$

Example 20.2 Calculate $\iint_R f(x,y)dA$ for

$f(x,y) = 1 - 6x^2 y$ and $R: 0 \le x \le 2, -1 \le y \le 1$

Solution: By Fubin's theorem

$$\iint_R f(x,y)dA = \int_{-1}^{1} \int_0^2 (1 - 6x^2 y)\, dx\, dy$$

$$= \int_{-1}^{1} \left(x - 2x^3 y \right) \Big|_{-1}^{2} dy$$

$$= \int_{-1}^{1} (2 - 16y)\, dy$$

$$= 2y - 8y^2 \Big|_{-1}^{1}$$

$$= (2 - 8) - (-2 - 8) = 4$$

Reversing the order of integration gives the same answer:

$$\int_{-1}^{1}\int_{0}^{2}(1-6x^2y)dy\,dx = \int_{0}^{2} y-3x^2y^2\Big|_{-1}^{1} dx$$

$$= \int_{0}^{2}\Big[(1-3x^2)-(-1-3x^2)\Big]dx$$

$$= \int_{0}^{2}\Big[1-3y^2+1+3x^2\Big]dx$$

$$= 2x\Big|_0^2 = 4\,.$$

20.1.2 How to determine the limits of Integration

The difficult part of evaluating a double integral can be finding the limits of integration. But there is a procedure to follow:

If we want to evaluate over a region R, integrating first with respect to y and then with respect to x, we take the following steps:

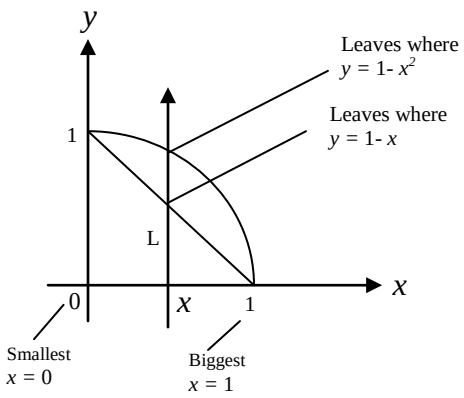
1. We imagine a vertical Line L cutting through in the direction of increasing y

2. We integrate from the y-value where L enters R to the y-value where L leaves R

3. We choose x-limits that include all the vertical lines that pass through R

Example 20.3 Change the order of integral $\int_{x=0}^{x=1}\int_{y=1-x}^{y=\sqrt{1-x^2}} f(x,y)dy\,dx$

To calculate the same double integral as an iterated integral with order of integration reversed consider (the figure), by using the above procedure, we have



$$\int_0^1 \int_{x=1-y}^{x=\sqrt{1-y^2}} f(x,y)\,dx\,dy$$

Example 20.4 Calculate $\iint_A \frac{\sin x}{x}\,dA$ where A is the triangle in the xy-plane bounded by the x-axis, the line y = x and the line y = 1.

Solution: $\int_0^1 \left(\int_0^x \frac{\sin x}{x}\,dy \right) dx$

$$= \int_0^1 \left(\frac{\sin x}{x} y \Big|_{y=0}^{y=x} \right) dx$$

$$= \int_0^1 \sin x\,dx = -\cos x \Big|_0^1 = -\cos 1 + \simeq .46$$

If we reverse the order of integration and try to calculate

$\int_0^1 \int_y^1 \frac{\sin x}{x} dx\, dy$, we can't evaluate it because we can't express $\int \frac{\sin x}{x}$ in terms of elementary functions.

PROBLEM

Evaluate the following integrals and sketch the region over which each integration takes place.

1. $\int_0^3 \int_0^2 (4 - y^2) dy\, dx$

2. $\int_0^3 \int_{-2}^0 (x^2 y - 2xy) dy\, dx$

3. $\int_0^{\pi} \int_0^x x \sin y\, dy\, dx$

4. $\int_0^{\pi} \int_0^{\sin x} y\, dy\, dx$

5. Find the value of the integral $\int_{10}^1 \int_0^{\frac{1}{y}} ye^{xy} dx\, dy$

6. Sketch the region of integration of $\int_0^2 \int_{x^2}^{2x} f(x, y) dy\, dx$ and express the integral as an equivalent double integral with order of integration.

Ans.: 1. 16, 2. 0, 3. $\frac{\left(4 + \pi^2\right)}{2}$, 4. $\frac{\pi}{4}$, 5. $9 - 9e$ & 6.

Keywords: Multiple Integrals, Double Integrals, Triple Integrals, Area, Volume

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Suggested Readings

Tom M. Apostol (2003). Calculus, Volume II Second Editions, Publishers, John Willey & Sons, Singapore.

Lesson-21

Triple Integration

21.1 Introduction

If F(x, y, z) is the function defined on a bounded region D in space (a solid ball or truncated cone, for example of something resembling a swiss cheese, or a finite union of such objects) then the integral of F over D defined in the following way.

We partition a rectangular region about D into rectangular cells by planes parallel to the co-ordinate planes, as shown in Fig.

The cells have dimensions Δx by Δy by Δz. We number the cells that lie inside D in some order $\Delta V_1, \Delta V_2, ..., \Delta V_n$,

choose a point (x_k, y_k, z_k) in each ΔV_k, and form the sum

$$S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k)\Delta V_k \text{ ---------- (21.1)}$$

If F is continuous and the bounding surface of D is made of smooth surfaces joined along continuous curves, then as $\Delta x, \Delta y$ and Δz all approach zero the sum S_n will approach all limit.

$$\lim S_n = \iiint_D F(x, y, z)\, dV$$

We call this limit the triple integral of F over D. The limit also exists for some discontinuous functions.

Triple integrals share many algebraic properties with double and single integrals. Writing F by $F(x,y,z)$ and G for $G(x,y,z)$, we have the following

1. $\iiint_D k\,F\,dV = k\iiint_V F\,dV$ $(any\,number\,k)$
2. $\iiint_D (F \pm G)dV = \iiint_D F\,dV \pm \iiint_D G\,dV$
3. $\iiint_D F\,dV \geq 0\ if\ F \geq 0\ in\ D$
4. $\iiint_D F\,dV \geq \iiint_D G\,dV\ \ if\ F \geq G\ on\ D$

If the domain D of a continuous function F is partitioned by smooth surface into a finite number of cells $D_1, D_2, ..., D_n$, then

5. $\iiint_D F\,dV = \iiint_{D_1} F\,dV + \iiint_{D_2} F\,dV + ... + \iiint_{D_n} F\,dV$

The triple integral Evaluation is hardly evaluated directly from its definition as a limit. Instead, one applies a three-dimensional version of Fubin's theorem to evaluate the integral by repeated single integrations.

For example, suppose we want to integrate a continuous function F(x, y, z) over a region D that is bounded below by a surface $z = f_1(x, y)$ above by the surface
$z = f_2(x, y)$, and on the side by a cylinder C parallel to the z – axis (Fig. 2). Let R denote the vertical projection of D onto the xy-plane enclosed by C. The integral of F over D is then evaluated as

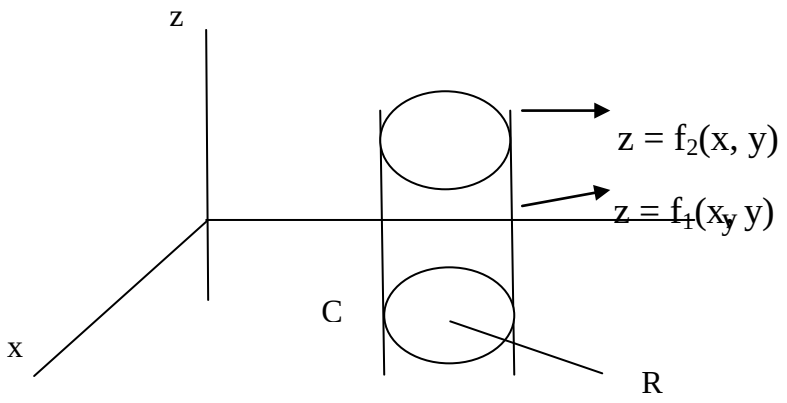

Fig.12

$$\iiint_D F(x,y,z)dV = \iint_R \left(\int_{f_1(x,y)}^{f_2(x,y)} F(x,y,z)dz \right) dy\,dx$$

or $$\iiint_D F(x,y,z)dV = \iint_R \int_{f_1(x,y)}^{f_2(x,y)} F(x,y,z)dz\,dy\,dx$$ ---------- (21.1)

If we omit the parenthesis .The z-limits of integration indicate that for every (x, y) in the region R, z may extend from the lower surface $z = f_1(x, y)$ to the upper surface $z = f_2(x, y)$. The y – and x - limits of integration have not given explicitly in Eq (21.1) but are to be determined in the usual way from the boundaries of R.

We will find the equation of the boundary of R by eliminating z between the two equations $z = f_1(x, y)$ and $z = f_2(x, y)$. This gives

$$f_2(x, y) = f_1(x, y),$$

an equation that contains no z and that defines the boundary of R in the xy - plane.

To give the z -limits of integration in any particular instance we may use a procedure like the one for double integrals. We imagine a line L through a point (x, y) in R and parallel to the z-axis. As z increases, the line enters D at $z = f_1(x, y)$ and leaves D at $z = f_2(x, y)$. These give the lower and upper limits of the integration with respect to z . The result of this integration is now a function of x and y alone, which we integrate over R, giving limits in the familiar way.

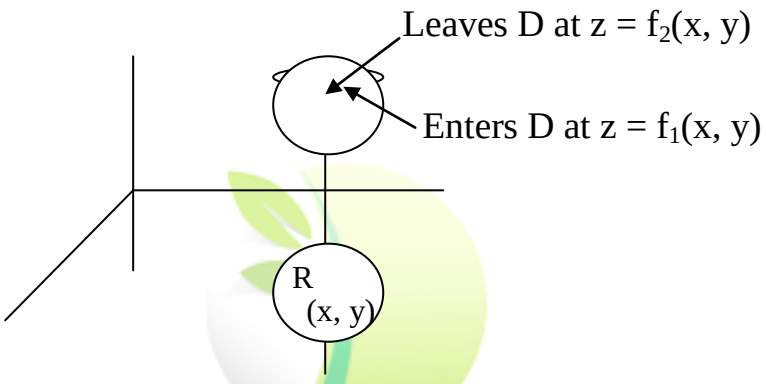

Fig. 12

Example 21.1 Find the volume enclosed between the two surfaces $z = x^2+3y^2$ and
$z = 8-x^2-y^2$.

Solution: The two surfaces intersect on the surface

$$x^2+3y^2 = 8-x^2-y^2$$

or $$x^2+2y^2 = 4$$

which is elliptic .

So the volume of the surface is

$$V = \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\, dy\, dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \left(8 - x^2 - y^2 - x^2 - 3y^2\right) dy\, dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \left(8 - 2\sqrt{x^2 - 4y^2}\right) dy\, dx$$

$$= \int_{-2}^{2} \int_{0}^{\sqrt{(4-x^2)/2}} 2\left(8 - 2\sqrt{x^2 - 4y^2}\right) dy\, dx$$

$$= \int_{-2}^{2} 2\left((8 - 2x^2)y - \frac{4}{3}y^3 \right)\Bigg|_{0}^{\sqrt{(4-x^2)/2}} dx$$

$$= \int_{-2}^{2} \left(2(8 - 2x^2)\sqrt{\frac{(4-x^2)}{2}} - \frac{8}{3}\left(\frac{(4-x^2)}{2}\right)^{\frac{3}{2}} \right) dx$$

$$= \frac{4\sqrt{2}}{3} \int_{-2}^{2} \left(4 - x^2\right)^{\frac{3}{2}} dx$$

$$= \frac{8\sqrt{2}}{3} \int_{0}^{2} \left(4 - x^2\right)^{\frac{3}{2}} dx$$

$$= 8\pi\sqrt{2}$$

As we know, there are sometimes two different orders in which the single integrations that evaluate a double integral may be worked (but not always). For triple integral there are sometimes (but not always) as many as six workable orders of integration. The next example shows an extreme case in which all six are possible.

Example 21.2 Each of the following integrals gives the volume of the solid shown

in Fig 3.

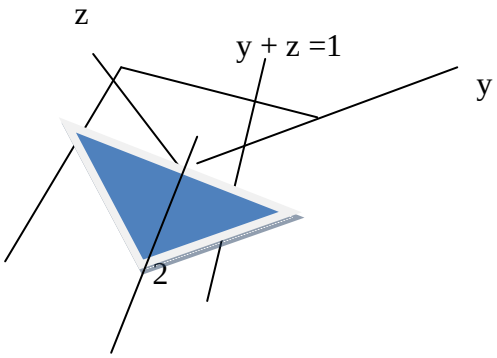

Fig 3.

$$(a)\int_0^1\int_0^1\int_0^2 dx\,dy\,dz \qquad (b)\int_0^1\int_0^{1-y}\int_0^2 dx\,dz\,dy$$

$$(c)\int_0^1\int_0^1\int_0^{1-z} dy\,dx\,dz \qquad (d)\int_0^2\int_0^1\int_0^{1-z} dy\,dz\,dx$$

$$(e)\int_0^1\int_0^2\int_0^{1-y} dz\,dx\,dy \qquad (f)\int_0^2\int_0^1\int_0^{1-y} dz\,dy\,dx$$

EXERCISES

1. Write six different iterated triple integrals for the volume of the rectangular solid in the first octant bounded by the co-ordinate planes and the planes x = 1, y = 2,
z = 3. Evaluate one of the integrals.

2. Write six different intersected triple integrals of the volume in the first octant enclosed by the cylinder $x^2 + z^2 = 4$ and the plane y = 3. Evaluate one of the integrals.

3. Write an iterated triple integrals in the order dz dy dx for the volume of the region bounded below by the xy-plane and above by the paraboloid $z = x^2 + y^2$ and lying inside the cylinder $x^2 + y^2 = 4$.

4. Rewrite the integral $\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} dz\,dy\,dx$ as an equivalent integrated integral in the order.

a) *dy dz dx* b) *dy dx dz* c) *dx dy dz* d) *dx dz dy* e) *dz dx dy*

Ans.: **1.** $\int_0^1\int_0^2\int_0^3 dz\,dy\,dx$, $\int_0^2\int_0^1\int_0^3 dz\,dx\,dy$, $\int_0^3\int_0^2\int_0^1 dx\,dy\,dz$, $\int_0^3\int_0^1\int_0^2 dy\,dx\,dz$, $\int_0^2\int_0^3\int_0^1 dx\,dz\,dy$, the value of each integral is 3, 2. $\int_0^3\int_0^2\int_0^{\sqrt{4-x^2}} dz\,dx\,dy$, $\int_0^2\int_0^3\int_0^{\sqrt{4-x^2}} dz\,dy\,dx$, $\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^3 dy\,dz\,dx$, $\int_0^2\int_0^{\sqrt{1-z^2}}\int_0^3 dy\,dz\,dx$, $\int_0^3\int_0^2\int_0^{\sqrt{4-z^2}} dx\,dz\,dy$, $\int_0^2\int_0^3\int_0^{\sqrt{1-x^2}} dx\,dy\,dz$. Value of each integral is 12π.

3. $4\int_0^2\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{x^2+y^2}} dz\,dy\,dx$ & 4.

Keywords: Triple integral, Fubini's theorem, volume

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Suggested Readings

Tom M. Apostol (2003). Calculus, Volume II Second Editions, Publishers, John Willey & Sons, Singapore.

Lesson 22

Area & Volume using Double and Triple Integration

22.1 Introduction

We have seen if we take f (x, y) = 1 in the definition of the double integral over a region in Eqn (20.2), is the partial sum reduce to

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k = \sum_{k=1}^{n} \Delta A_k,$$

and give area of the region as $n \to \infty$. In that case $\Delta x, \Delta y$ approach zero. In this case we define the area on a rectangular region R to be the limit

$$Area = \lim \sum \Delta A_k = \iint_R dA \qquad (22.1)$$

Example 22.1 Find the area of the region R bounded by y = x and y = x^2 in the first quadrant.

Solution: The area of the region is

$$\int_0^1 \int_{x^2}^{x} dy\, dx = \int_0^1 (x - x^2)dx$$

$$= \frac{x^2}{2} - \frac{x^3}{3}\Big|_0^6 = \frac{1}{6}$$

Example 22.2 Find the area of the region R enclosed by the parabola y = x^2 and the line y = x + 2.

Solution: $x^2 = x+2 \Rightarrow x^2 - x - 2 = 0$

$x^2 - 2x + x - 2 = 0 \quad i.e., \quad x(x-2)+1(x-2) = 0$

$(x+1)(x-2) = 0$

$x = -1, 2$

Hence the area $A = \int_{-1}^{2}\int_{x^2}^{x+2} dy\,dx$

$$= \int_{-1}^{2} y\Big|_{x^2}^{x+2} dx$$

$$= \int_{-1}^{2} (x+2-x^2)dx$$

$$= \frac{x^2}{2} + 2x - \frac{x^3}{3}\Big|_{-1}^{2}$$

$$= \left(2+4-\frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)$$

$$= 2+4-\frac{8}{3} - \frac{1}{2} - \frac{1}{3} + 2$$

$$= 8 - \frac{16+3+2}{6}$$

$$= 8 - \frac{7}{2} = \frac{9}{2}$$

Solution:

For order of integration reversed, draw a horizontal lin L_2. It enters at $x = \frac{y}{2}$, leaves at $x = \sqrt{y}$. To include all such lines we let y to n from y = 0 to y = 4. The integral is

$$\int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} f(x,y)dx\,dy$$

22.1.1 Changing to Polar Coordinates.

When we define the integral of a function f(x, y) over a region R we divide R with rectangles, and their areas easy to compute. But when we work in polar coordinates, however it is more natural to subdivide R into 'polar rectangles' we can find the double integral in polar form as.

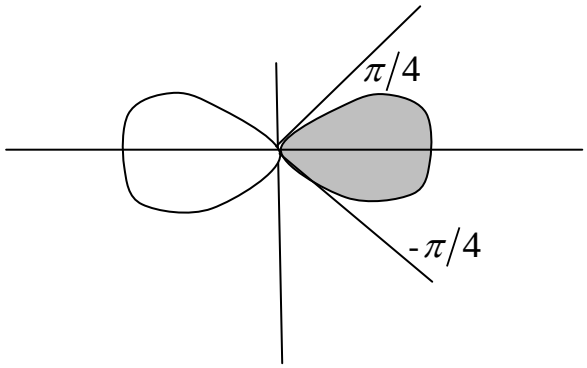
$$\iint F(r,\theta)\,dA = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=f_1(\theta)}^{r=f_2(\theta)} F(r,\theta)\,r\,dr\,d\theta \quad \text{---------- (22.2)}$$, give running numbers.

Where the function $F(r,\theta)$ is defined over a region R bounded by the areas $\theta=\alpha$, $\theta=\beta$ and the continuous curve $r=f_1(\theta)$, $r=f_2(\theta)$.

If $F(r,\theta)\equiv 1$ the constant function whose value is one, then the value over R is the areas of R (which agrees our earlier definition). Thus

Area of R = $\iint_R r\,dr\,d\theta$

Example 22.3 Find the area enclose by the lemniscate $r^2 = 2a^2\cos 2\theta$.



The area of the right-hand half to be

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{0}^{\sqrt{2a\cos^2 2\theta}} r\,dr\,d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{r^2}{2}\Big|_{r=0}^{r=\sqrt{2a\cos^2 2\theta}} d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} a^2 \cos 2\theta\, d\theta$$

$$= \frac{a^2}{2}\sin 2\theta\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{a^2}{2}[1-(-1)]$$

$$= a^2$$

The total area is therefore $2a^2$.

22.2 Volume using Triple Integral

If $F(x, y, z) \equiv 1$ is the constant function whose volume is one, then the sums in Eq (1) reduce to $S_n = \sum_{k=1}^{n} 1.\Delta V_K = \sum_{k=1}^{n} \Delta V_K$

As $\Delta_x, \Delta_y, \Delta_z$ all approaches zero, the cells ΔV_k become smaller and we need more cells to fill up D. We therefore define the volume of D to be the triple integral of the constant function f(x, y, z) = 1 over D.

Volume of D = $\lim \sum_{k=1}^{n} \Delta V_k = \iiint_D dV$.

The triple integral Evaluation is hardly evaluated directly from its definition as a limit. Instead, one applies a three-dimensional version of Fubin's theorem to evaluate the integral by repeated single integrations.

22.3 Integrals in Cylindrical and Spherical Coordinates

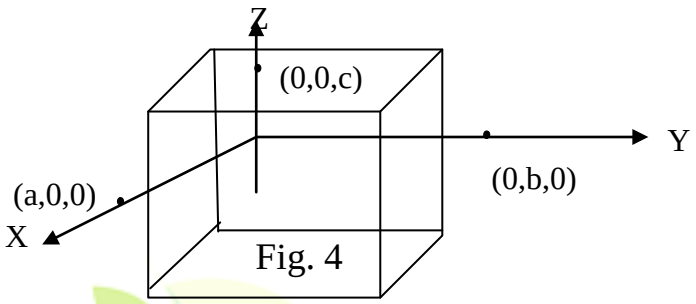


Fig. 4

Fig. 4 shows a system of mutually orthogonal coordinates axes OX, OY, OZ. The Cartesian coordinates of a point P(x, y, z) in the space may be read from the coordinates axes by passing planes through P perpendicular to each axis. The points on the x-axis have their y- and z- ordinates both zero. Points in a plane perpendicular to the z-axis, say, all have the same z - coordinate. Thus of the points in the plane perpendicular to the z- axis and 5 units above the xy-plane all have coordinates of the form (x,y,5) . We can write z = 5 as an evaluation for this plane. The three planes x = 2, y = 3, z = 5 intersect in the point P (2, 3, 5). The points of the yz- plane are obtained setting x = 0. The three coordinates planes x = 0, y = 0, z = 0 divide the space into eight cells, called octants. The octant in which all three coordinates are positive is the first octant, but there is no conventional numbering of the remaining seven octants.

Example 2. Describe the set of points P(x, y, z) whose Cartesian coordinates satisfy the simultaneous equation $x^2 + y^2 = 4$, z = 3.

Solution: The points all horizontal plane z = 3, and in this plane they lie in this cirle $x^2 + y^2 = 4$.Thus we may describe the set of the circle in the plane $x^2 + y^2 = 4$ in the plane z = 3.

22.3.1 Cylindrical Coordinates

It is frequently convenient to use cylindrical coordinates (r,θ,z) to locate a point in space. These are just the polar coordinates (r,θ) used instead of (x, y) in the plane z = 0, coupled with the z- coordinates. Cylindrical and Cartesian coordinate are therefore related by the following equations : Equations relating cartesian and cylindrical coordinates.

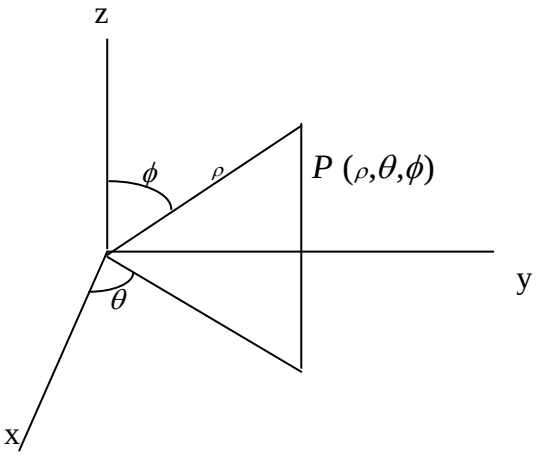
$x = r\cos\theta$ $\qquad$ $r = x^2 + y^2$

$y = r\sin\theta$ $\qquad$ $\tan\theta = \frac{y}{x}$

$z = z$

22.3.2 Spherical Coordinates

Spherical coordinates are useful when there is a center of symmetry that we can take as the origin. The spherical coordinates (ρ,φ,θ) are shown the first coordinates $\varphi = |OP|$ is the distance from the origin to the point. It is never negative. The equation $\varphi =$ constant describes the surface of the sphere of radius φ with centre O.



The second spherical coordinate ϕ, is the angle measured down from the z-axis to the line OP. The equation $\rho =$ constant describes cone with vertex at O, axis OZ and generating angle ϕ, provide we broaden our interpretation of the word "cone" to include the xy- plane for which $\phi = \frac{\pi}{2}$ and cones the generation angles greater than $\frac{\pi}{2}$.

The third spherical coordinates θ is the same as the angle θ in cylindrical coordinates, namely, the angle from the xz-plane the plane through P and the z-axis.

22.3.3. Coordinate Conversion Formulas

We have the following relationships between these Cartesian (x, y, z), cylindrical (r, θ, z), *and spherical* (ρ, φ, θ)

Polar to Rectangular	Spherical to Cylindrical	Spherical to Rectangular
$x = r\cos\theta$	$r = \rho\sin\phi$	$x = \rho\sin\phi\cos\theta$
$y = r\sin\theta$	$r = \rho\cos\phi$	$y = \rho\sin\phi\sin\theta$
$z = z$	$\theta = \theta$	$z = \rho\cos\theta$

Volume : $\iiint dx\,dy\,dz = \iiint dz\,r dr\,d\theta = \iiint \rho^2 \sin\theta\, d\rho\, d\phi\, d\theta$

Exercises

1. Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 1$

2. Find the area of the region R bounded by $y = x$ and $x = y^2$ in the first quadrant.

3. Find the volume of the solid in the first octant bounded by the paraboloid. $z = 36 - 4x^2 - 9y^2$

4. Find the volume of the solid enclosed between the surfaces $x^2 + y^2 = 9^2$ and $x^2 + z^2 = 9^2$.

5. The volumes of the tetrahedron bounded by the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ and the coordinate planes.

6. The volume in the first octant bounded by the planes $x + z = 1$, $y + 2z = 2$.

7. The volume of the wedge cut from the cylinder $x^2 + y^2 = 1$ and the plane $z = y$ above and plane below.

8. The volume of the region in the first octant bounded by the coordinate planes, above by the cylinder $x^2 + z = 1$ and on the right by the paraboloid $y = x^2 + z^2$

(Hint: Integrate first with respect to y)

Ans.: 1. ,2. , 3. 27π, 4. $\frac{16a^3}{3}$, 5. $\frac{1}{6}|abc|$, 6. $\frac{2}{3}$, 7. $\frac{2}{3}$ & 8. $\frac{2}{7}$

Keywords: Area, Volume, Double Integral, Triple Integral

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Suggested Readings

Tom M. Apostol (2003). Calculus, Volume II Second Editions, Publishers,John Willey & Sons, Singapore.

Lesson 23

Gamma Function

23.1 Introduction: We shall define a function known as the gamma function, $\Gamma(x)$ which has the property that $\Gamma(n) = (n-1)!$ for every positive integer n. It may be regarded then a generalization of factorial n to apply to values of the variable which are not integer. The function is defined in terms of an improper integral. This integral cannot be evaluated in terms of the elementary functions. It has great importance in analysis and in applications.

Definition 23.1 The Gamma Function: The gamma function is defined by the improper integral

$$\Gamma(\lambda+1) = \int_0^{+\infty} e^{-t} t^{\lambda}\, dt \text{ ---------- (23.1)}$$

which converges for all $\lambda > -1$

To deduce some of the properties of the gamma function, let us integrate Eq. (23.1) by parts:

$$\int_0^{+\infty} e^{-t} t^{\lambda}\, dt = \lim_{R\to+\infty} \int_0^{R} e^{-t} t^{\lambda}\, dt$$

$$= \lim_{R\to+\infty} \left[-e^{-t} t^{\lambda} \Big|_0^R + \lambda \int_0^R e^{-t} t^{\lambda}\, dt \right]$$

$$= \lim_{R\to+\infty} \left[\frac{-R^{\lambda}}{e^R} + 0 \right] + \lambda \int_0^{+\infty} e^{-t} t^{\lambda-1}\, dt$$

$$= \lambda \int_0^{+\infty} e^{-t} t^{\lambda-1}\, dt$$

$$\text{i.e. } \Gamma(\lambda+1) = \lambda\Gamma(\lambda) \text{ ---------- (23.2)}$$

If we let $\lambda = 0$ in Eq 1. these results

$$\Gamma(1) = \lambda\int_0^\infty e^{-t}\,dt = -e^{-t}\Big|_0^\infty = 1$$

Using Eq 23.2, we obtain

$$\Gamma(2) = 1.\Gamma(1) = 1$$
$$\Gamma(3) = 2.\Gamma(2) = 2!$$
$$\Gamma(4) = 3.\Gamma(3) = 3! \text{ ---------- (23.3)}$$

The equations above represent another important property of the gamma function. $1+\lambda$ is a positive integer.

$$\Gamma(\lambda+1) = \lambda! \text{ ---------- (23.4)}$$

It is interesting to note that $\Gamma(\lambda)$ is defined for all λ except $\lambda = 0, -1, -2, ...$by the functional equation $\Gamma(\lambda+1) = \lambda\Gamma(\lambda)$; infact, we need to know $\Gamma(\lambda)$ only for $1 \le \lambda \le 2$ to compute $\Gamma(\lambda)$ for all real values of λ. Fig 1. Illustrates the graph $\Gamma(\lambda)$

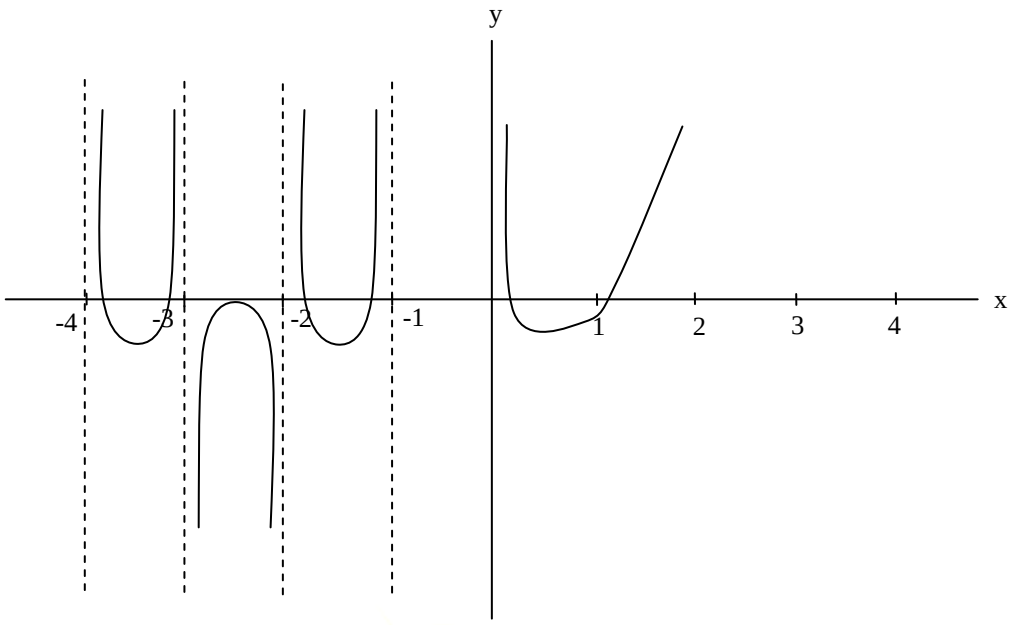

Fig 1. $\Gamma(\lambda)$ the Gamma function

Certain constants related to $\Gamma(x)$. We shall show that $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$. In order to do this, we compute first the so-called probability integral.

Theorem 23.1. $\int_0^{+\infty} e^{-x^2}\,dx=\frac{1}{2}\sqrt{\pi}$

To prove this, consider the double integral of $e^{-x^2-y^2}$ over two circular sectors D_1 and D_2 and the Square S indicated in Fig 2.

Since the integral is positive, we have

$$\iint_{D_1} < \iint_{S} < \iint_{D_2} \qquad (24.1.5)$$

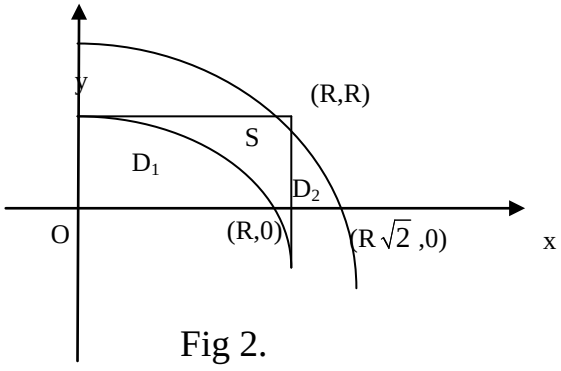

Fig 2.

Now evaluate these integrals by iterate integrals, the centre one in rectangular coordinates, and other two in polar coordinates:

$$\int_0^R e^{-r^2} r\,dr \int_0^{\frac{\pi}{2}} d\theta < \int_0^R e^{-x^2} dx \int_0^R e^{-y^2} dy < \int_0^{R\sqrt{2}} e^{-r^2} r\,dr \int_0^{\frac{\pi}{2}} d\theta$$

$$\frac{\pi}{4}\left(1-e^{-R^2}\right) < \left(\int_0^R e^{-x^2} dx\right)^2 < \frac{\pi}{4}\left(1-e^{-2R^2}\right)$$

Now let $R \to \infty$, then

$$\left(\int_0^{+\infty} e^{-x^2} dx\right)^2 = \frac{\pi}{4}$$

i.e., $\int_0^{+\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

Theorem 23.2. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Now, $\Gamma\left(\frac{1}{2}\right) = \int_0^{+\infty} e^{-t} t^{\frac{1}{2}-1} dt = \int_0^{+\infty} e^{-t} t^{-\frac{1}{2}} dt$

$$= 2\int_0^{+\infty} e^{-y^2} dy = \sqrt{\pi} \quad \text{set t = } y^2$$

Example 23.1 Evaluate the integral $\int_0^{+\infty} x^{\frac{5}{4}} e^{-\sqrt{x}} dx$

Solution: Set $x = t^2$, $dx = 2tdt$

$$\int_0^{+\infty} x^{\frac{5}{4}} e^{-\sqrt{x}} dx = 2\int_0^{+\infty} t^{\frac{7}{2}} e^{-t} dt = 2\Gamma\left(\tfrac{9}{2}\right)$$

From the recursive relation (2) , we obtain

$$\Gamma(\tfrac{9}{2}) = \frac{7}{2}.\frac{5}{2}.\frac{3}{2}\Gamma(\tfrac{3}{2}) = \frac{105}{8}\frac{\sqrt{\pi}}{2}$$

Finally, the volume of the integral is

$$\int_0^{+\infty} x^{\frac{5}{4}} e^{-\sqrt{x}} dx = 2 \times \frac{105}{8} \times \frac{\sqrt{\pi}}{2} = \frac{105\sqrt{\pi}}{8}$$

Example 23.2 Express the product

$f(r) = r(r+h)(r+2h).....[r+(n-1)h]$ as a quotient of gamma functions.

Solution: We have

$$f(r) = \left(\frac{r}{h}\right)\left(\frac{r}{h}+1\right)\left(\frac{r}{h}+2\right).....\left(\frac{r}{h}+(n-1)h\right)h^n$$

$$= h^n \frac{\Gamma\left(\frac{r}{h}+1\right)}{\Gamma\left(\frac{r}{h}\right)} \frac{\Gamma\left(\frac{r}{h}+2\right)}{\Gamma\left(\frac{r}{h}+1\right)} .... \frac{\Gamma\left(\frac{r}{h}+n\right)}{\Gamma\left(\frac{r}{h}+n-1\right)}$$

$$= \frac{\Gamma\left(\frac{r}{h}+n\right)}{\Gamma\left(\frac{r}{h}\right)}.h^n$$

obtained by the recursion Eq. 2 with $\lambda = \frac{r}{h}$

Some special cases of the result of Example 2 are interesting. For particular case, set r = 1 and h = 2. Then

$$1.3.5....(2n-1)=\frac{2^n\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})}$$

But $\frac{1}{2}\Gamma(\frac{1}{2})=\Gamma(\frac{3}{2})=\frac{\sqrt{\pi}}{2}$.

Hence

$$1.3.5....(2n-1)=\frac{2^n\Gamma(n+\frac{1}{2})}{\sqrt{\pi}}$$

However,

$$1.3.5....(2n-1)=1.3.5....(2n-1)\frac{2.4.6.....2n}{2.4.6.....2n}$$

$$=\frac{(2n)!}{2^n n!}$$

Now combining the two equations above , we get

$$\Gamma\left(n+\tfrac{1}{2}\right)=\frac{(2n)!}{2^n n!}\times\frac{\sqrt{\pi}}{2^n}=\frac{(2n)!}{2^{2n}n!}\sqrt{\pi}$$

for n = 1, 2, ……

Other expressions for $\Gamma(x)$

Theorem 23.3. $\Gamma(x)=r^x\int_0^{+\infty}e^{-rt}t^{x-1}\,dt,\ r>0,\ x>0$

This follows from the definition

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1}\, dt,\ set\ rt = y$$

Theorem 23.4. $\Gamma(x) = 2\int_0^{+\infty} e^{-t^2} t^{2x-1}\, dt$

Proof: Set $t^2 = y$

Extension of definition

Definition : For n =1,2,….

$$\Gamma(x) = \frac{\Gamma(x+n)}{x(x+1)(x+2).....(x+n-1)}, -n < x < -n+1$$

Thus we have defined $\Gamma(x)$ for all x except x = 0,-1,-2,…. Observe that when n =1 the right hand side of (6) depends on the values of $\Gamma(x)$ in the interval 0<x<1. It is clear that $\Gamma(x)$ has been defined for negative x in such a way that equation

$$\Gamma(x+1) = x\Gamma(x) \text{ for } x \neq 0, -1, -2, ... \text{---------(23.6)}$$

Example 4. Compute $\Gamma\left(\frac{1}{2}\right)$

From equation (7), we have

$$\Gamma\left(-\tfrac{1}{2}+1\right) = -\tfrac{1}{2}\Gamma\left(-\tfrac{1}{2}\right)$$

i.e., $\Gamma\left(\frac{1}{2}\right) = -\frac{1}{2}\Gamma\left(-\frac{1}{2}\right)$

i.e., $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$

Exercise

Evaluate each integral

1. $\int_0^{+\infty} \sqrt{x}\, e^{-x} dx$

2. $\int_0^{+\infty} x^2\, e^{-x^2} dx$

3. $\int_0^{+\infty} x^{-4}\, e^{-\sqrt{x}} dx$

4. $\int_0^{+\infty} (1-x)^3\, e^{-\sqrt{x}} dx$

5. $\int_0^{+\infty} x^3\, e^{-\sqrt{x}} dx$

6. Show that the improper integral $\int_0^{+\infty} e^{-t} t^x dt$ converges for $x > -1$ and diverges for $x \le -1$.

7. Compute $\int_0^1 \frac{dx}{\sqrt{x \ln\left(\frac{1}{x}\right)}}$

8. Evaluate $\int_0^{\infty} 2^{-9x^2} dx$ using gamma function (Hint : $2^{-9x^2} = e^{-9x^2 \ln 2}$)

Ans.: 1. $\Gamma\left(\frac{3}{2}\right)$ *or* $\frac{\sqrt{\pi}}{2}$, 2. 6, 3. ∞, 4. -9394, 5. $2\times 7!$, 6. ,7. $\sqrt{2\pi}$ & 8. $\frac{1}{6}\sqrt{\frac{\pi}{\ln 2}}$

Keywords: Gamma Function, Convergence of Integral, Factorial Function

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Suggested Readings

Tom M. Apostol (2003). Calculus, Volume II Second Editions, Publishers, John Willey & Sons, Singapore.

Lesson 24

The Beta Function

24.1 Introduction

In this Lesson we shall introduce a useful function of two variables known as beta function. Its usefulness is considerably overshadowed by that of gamma function. In fact, we shall show that it can be evaluated in terms of the latter function. As consequence, it would be unnecessary to introduce it as a new function. Since it occurs so frequently in analysis, a special designation for it is accepted.

Definition 2.2

For *x*, *y* positive we define the Beta function by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt.$$

Using the substitution $u = 1 - t$ it is easy to see that

Theorem 24.1. $B(x,y) = B(y,x)$.

Here we say the beta function is symmetric.

To evaluate the Beta function we usually use the Gamma function. To find their relationship, one has to do a rather complicated calculation involving change of variables (from rectangular into tricky polar) in a double integral.

When x and y are positive integers, it follows from the definition of the gamma function Γthat:

$$\mathrm{B}(x,y) = \frac{(x-1)!\,(y-1)!}{(x+y-1)!}$$

Theorem 24.2. For $0 < x < \infty, 0 < y < \infty$,

$$\beta(x,y) = \int_{0+}^{\infty} (\sin t)^{2x-1} (\cos t)^{2y-1} dt$$

To prove this set $t = \sin^2 u$ in the integral.

$$\mathrm{B}(x,y) = 2\int_0^{\pi/2} (\sin\theta)^{2x-1} (\cos\theta)^{2y-1}\, d\theta, \qquad \mathrm{Re}(x) > 0,\ \mathrm{Re}(y) > 0$$

Theorem 24.3. For $0 < x < \infty, 0 < y < \infty$,

$$\beta(x,y) = \int_{0+}^{\infty} \frac{t^{x-1}}{(1+t)^{x+y}} dt$$

Here the change of variable $t = u(1+u)^{-1}$ suffices.

It has many other forms, including:

Theorem. For $0 < x < \infty, 0 < y < \infty$

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$$

Proof : When x and y are arbitrary positive numbers, the proof proceeds as follows. From the double integral of the nonnegative function $t^{2x-1}u^{2y-1}e^{-t^2-u^2}$ over the three regions D_1, D_2 and S of figure 1 of Lesson 23. Now, however, t and u are the variables, however, t and u are the variables x and y positive constants. We have relation (23.5) of Lesson 23 as before. Again we evaluate the central double integral by iteration in rectangular coordinates: the other two, in polar coordinates:

$$\int_0^{\frac{\pi}{2}} \cos^{2x-1}\theta \sin^{2y-1}\theta\, d\theta \int_0^R e^{-r^2} r^{2x+2y-1}\, dr < \int_0^R t^{2x-1}e^{-t^2}dt \int_0^R u^{2y-1}e^{-u^2}du$$

$$< \int_0^{\frac{\pi}{2}} \cos^{2x-1}\theta \sin^{2y-1}\theta\, d\theta \int_0^{R\sqrt{2}} e^{-r^2} r^{2x+2y-1}\, dr$$

Now, if we let R become infinite and use Theorems 23.4 and 24.3, we obtain

$$\frac{1}{2}B(y,x)\frac{1}{2}\Gamma(x+y) = \frac{\Gamma(x)}{2}\frac{\Gamma(y)}{2}, 0<x, 0<y$$

This completes the proof of the theorem.

Example 24.1 Evaluate $\int_0^1 x^4(1-x)^3 dx$

Solution: $\int_0^1 x^4(1-x)^3 dx = \int_0^1 x^{5-1}(1-x)^{4-1}dx = B(5,4) = \frac{\Gamma(5)\Gamma(4)}{\Gamma(9)} = \frac{1}{280}$

Example 24.2 Evaluate $\int_0^1 \frac{1}{\sqrt[3]{x^2(1-x)}} dx$

Solution: $\int_0^1 \frac{1}{\sqrt[3]{x^2(1-x)}}dx = \int_0^1 x^{\frac{1}{3}-1}(1-x)^{\frac{2}{3}-1}dx = \beta(\frac{1}{3},\frac{2}{3}) = \frac{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})}{\Gamma(1)} = \Gamma(\frac{1}{3})\Gamma(\frac{2}{3})$

Example 24.3 Evaluate $\int_0^1 \sqrt{x}.(1-x)dx$

Solution: $\int_0^1 \sqrt{x}(1-x)dx = \int_0^1 x^{\frac{3}{2}-1}(1-x)^{2-1}dx = \beta\left(\frac{3}{2},2\right) = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(2)}{\Gamma\left(\frac{7}{2}\right)}$

$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}$

$\Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}$

$\Gamma\left(\frac{7}{2}\right) = \frac{15}{8}\sqrt{\pi}$

Thus $\int_0^1 \sqrt{x}.(1-x)dx = \frac{4}{15}$

Example 24.4 Given $\int_0^\infty \frac{x^{q-1}}{1+x}dx = \frac{\pi}{\sin n\pi}$, show that $\Gamma(q)\Gamma(1-q) = \frac{\pi}{\sin n\pi}$

Proof: We know,

for $0 < x < \infty, 0 < y < \infty,$

$$\beta(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}dt$$

$$\int_0^\infty \frac{x^{q-1}}{(1+x)^{q+(1-q)}}dx = \beta(q,1-q)$$

$$= \frac{\Gamma(q)\Gamma(1-q)}{\Gamma(1)} = \Gamma(q)\Gamma(1-q)$$

Example 24.5 Evaluate $I = \int_0^\infty \frac{dx}{(1+x^4)}$

Solution: Let $x^4 = t, 4x^3dx = dt$

$$I = \frac{1}{4}\int_0^\infty \frac{t^{-\frac{3}{4}}}{1+t}\,dt = \frac{1}{4}\int_0^\infty \frac{t^{\frac{3}{4}-1}}{1+t}\,dt = \frac{1}{4}\frac{\pi}{\sin\frac{1}{4}\pi} = \frac{\pi}{4\times\frac{1}{\sqrt{2}}}$$

Exercises

1. $\int_0^1 t^3\,(1-t)^3 dt$
2. $\int_0^1 \sqrt[3]{t(1-t)}\,dt$
3. $\int_{0+}^1 (1-\frac{1}{t})^{\frac{1}{3}} dt$
4. $\int_0^{\frac{\pi}{2}} \sqrt{\tan x}\,dx$
5. $\int_0^{\frac{\pi}{2}} (\sin 2x)^{\frac{1}{4}} dx$
6. $\int_{0+}^\infty \frac{1}{\sqrt{t}(1+t)}\,dt$
7. $\int_{0+}^\infty \frac{t\,dt}{(1+t)^3}$
8. $\int_{0+}^\infty \frac{dt}{(1+t)^2\sqrt{1+(1/t)}}$
9. $\int_{0+}^{\frac{\pi}{2}} (\sin 2x)^{2t-1}\,dx \qquad 0 < t < \infty$

Keywords: Gamma Function, Beta Function, Polar Coordinate.

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Suggested Readings

Tom M. Apostol (2003). Calculus, Volume II Second Editions, Publishers, John Willey & Sons, Singapore.

Lesson 25

Introduction

In this lesson we introduce basic concepts of theory of ordinary differential equations. Formation of the differential equation from a given family of curves is explained. Different types of solutions are defined. The given definitions are supplemented by some simple examples.

25.1 Differential Equations

An equation involving derivatives or differentials of one or more dependent variables with respect to one or more independent variables is called a differential equation. An ordinary differential equation of order n is defined by the relation

$$F(t, x, x^{(1)}, x^{(2)}, \ldots, x^{(n)}) = 0 \tag{25.1}$$

where $x^{(n)}$ stands for the nth derivative of unknown function $x(t)$ with respect to the independent variable t. For example

$$\frac{d^4x}{dt^4} + \frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^5 = e^t \tag{25.2}$$

$$\frac{dx}{dt} = x + \sin x. \tag{25.3}$$

25.1.1 Order of a Differential Equation

The order of a differential equation is referred to the highest order derivative involved in the differential equation. For example, the order of the differential Equation (25.2) is four.

25.1.2 Degree of Differential Equation

The degree of a differential equation is the degree of the highest order derivative which occurs in it; after the differential equation has been made free from radicals and fractions as far as derivatives are concerned, e.g. in differential Equation (25.2), the degree is one.

25.1.3 Linear and Nonlinear Differential Equation

A differential equation is called linear if (a) every dependent variable and every derivative involved occurs in first degree only, and (b) no product of dependent variables and/or derivatives occur. A differential is not linear is called nonlinear. For examples, Equation (25.2) is linear and (25.3) is nonlinear.

25.2 Solution of a Differential Equation

Any relation between the dependent and independent variables, when substituted in the differential equation, reduces it to an identity is called a solution of differential equation. For example, $y = e^{2x}$ is a solution of $y' = 2y$.

25.2.1 Example

Show that $y = A/x + B$ *is solution of*

$$y'' + \left(\frac{2}{x}\right) y' = 0$$

Solution: We have the differential equation

$$y'' + \left(\frac{2}{x}\right) y' = 0. \tag{25.4}$$

Also given that

$$y = A/x + B. \tag{25.5}$$

Differentiating (25.5) w.r.t. x

$$y' = -A/x^2. \tag{25.6}$$

Differentiating (25.6) w.r.t. x

$$y'' = 2A/x^3. \tag{25.7}$$

Substituting (25.6) and (25.7) into (25.4), we have

$$\frac{2A}{x^3} - \frac{2A}{x^3} = 0.$$

25.2.2 Complete, Particular and Singular Solutions

Let

$$F(t, x, x^{(1)}, x^{(2)}, \ldots, x^{(n)}) = 0 \tag{25.8}$$

be an n-th odder differential equation.

- A solution of (25.8) containing n independent constants is called *general solution*.
- A solution of (25.8) obtained from a general solution by giving particular value to one or more of the n independent arbitrary constants is called *particular solution*.
- A solution which cannot be obtained from any general solution by any choice of the n independent arbitrary constants is called *singular solution*.

25.3 Formation of Differential Equations

An n-parameter family of curves is a set of relations of the form $\{(x, y) : f(x, y, c_1, c_2, ..., c_n) = 0\}$, where f is real valued function of $x, y, c_1, c_2, ..., c_n$ and each c_i $(i = 1, 2, ...n)$ ranges over an interval of real values.

Suppose we are given a family of curves containing n arbitrary constants. Then by differentiating it successively n times and eliminating all arbitrary constants from the $(n + 1)$ equations we obtain an nth order differential equation whose solution is the given family of curves. We now illustrate the procedure of forming differential equations with the help of some examples.

25.4 Example Problems

25.4.1 Problem 1

Find the differential equation of the family of curves $y = e^{mx}$, where m is an arbitrary constant.

Solution: We have the family of curves

$$y = e^{mx}. \tag{25.9}$$

Differentiating (25.9) w.r.t x, we get

$$y' = me^{mx}. \tag{25.10}$$

Now, we eliminate m from (25.9) and (25.10) and using $m = \log_e y$, we obtain the required differential equation as

$$y' = y \log_e y.$$

25.4.2 Problem 2

Obtain the differential equation satisfied by the family of circles $x^2 + y^2 = a^2$, where a is an arbitrary constant.

Solution: The family of circles is given as

$$x^2 + y^2 = a^2. \tag{25.11}$$

Differentiating (25.11) w.r.t x, we get

$$x + yy' = 0,$$

which is the required differential equation.

25.4.3 Problem 3

Obtain the differential equation satisfied by $xy = ae^x + be^{-x} + x^2$, where a and b are an arbitrary constant.

Solution: Given family of curves

$$xy = ae^x + be^{-x} + x^2. \tag{25.12}$$

Differentiating (25.12) w.r.t x, we get

$$xy' + y = ae^x - be^{-x} + 2x, \tag{25.13}$$

Differentiating (25.14) w.r.t x and using (25.14), we get

$$xy'' + 2y' = (xy - x^2) + 2, \tag{25.14}$$

which is the required differential equation.

Remark: *From the above examples we observed that the number of arbitrary constants in a solution of a differential equation depends upon the order of the differential equation and is the same as its order. Hence a general solution of an nth order differential equation will contain n arbitrary constant.*

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Lesson 26

Differential Equation of First Order

In this lesson we present solution techniques of differential equations of first order and first degree. We shall mainly discuss differential equation of variable separable form, homogeneous equations and equations reducible to homogeneous form.

There are two standard forms of differential equations of first order and first degree, namely,

$$\frac{dy}{dx} = f(x, y) \qquad \text{or} \qquad Mdx + Ndy = 0$$

Here M and N are functions of x and y, or constants. We discuss here some special forms of these equations where exact solution can easily be obtained.

26.1 Separation of Variables

If in a differential equation, it is possible to get all the functions x and dx to one side and all the functions of y and dy to the other, the variables are said to be separable. In other words if a differential equation can be written in the form $F(x)dx + G(y)dy = 0$, we say variables are separable and its solution is obtained by integrating the equation as

$$\int F(x)dx + \int G(y)dy = c,$$

where c is a integration constant.

26.2 Example Problems

26.2.1 Problem 1

Solve $\frac{dy}{dx} = e^{x+y} + x^2e^y$.

Solution: For separating variables, we rewrite the given equation as

$$e^{-y}dy = (e^x + x^2)dx.$$

Integrating the above equation we have

$$-e^{-y} = e^x + x^3/3 + c,$$

where c is an arbitrary constant.

26.2.2 Problem 2

Solve $3e^x \tan y dx + (1 - e^x) \sec^2 y dy = 0$.

Solution: Separating the variables, we get

$$\frac{3e^x}{1 - e^x} dx + \frac{\sec^2 y}{\tan y} dy = 0.$$

Integration gives

$$-3 \log(1 - e^x) + \log(\tan y) = \log c,$$

where c is an arbitrary constant.

26.3 Equations Reducible to Separable Form

Differential equation of the form

$$\frac{dy}{dx} = f(ax + by + c) \quad or \quad \frac{dy}{dx} = f(ax + by)$$

can be reduced by the substitution $ax + by + c = v$ or $ax + by = v$ to an equation in which variables can be separated.

26.3.1 Example

Solve $\frac{dy}{dx} = \sec(x + y)$.

Solution: Let, $x + y = v$ so that

$$\frac{dy}{dx} = \frac{dv}{dx} - 1. \tag{26.1}$$

Using (26.1), the given differential equation becomes

$$\frac{dv}{dx} = \sec v + 1. \tag{26.2}$$

This equation is of separable form. Thus we have

$$dx = \frac{1}{\sec v + 1} dv \quad \Rightarrow \quad dx = \frac{2\cos^2(v/2) - 1}{1 + 2\cos^2(v/2) - 1} dv$$

Further simplifications gives

$$dx = \left(1 - \frac{1}{2}\sec^2(v/2)\right) dv$$

Integrating and substituting the value of v, we obtain $y - \tan\frac{1}{2}(x + y) = c$.

26.4 Homogeneous Differential Equation

A differential equation of first order and first degree is said to be homogeneous if it can be put in the form

$$\frac{dy}{dx} = f(y/x).$$

These equations can be solved by letting $y/x = v$ and differentiating with respect to x as

$$v + x\frac{dv}{dx} = f(v) \quad \Rightarrow \quad x\frac{dv}{dx} = f(v) - v.$$

Then, separating variables, we have

$$\frac{dx}{x} = \frac{dv}{f(v) - v}$$

Integrating the above equation we obtain

$$\log x + c = \int \frac{dv}{f(v) - v},$$

where c is an arbitrary constant. The solution is obtained by replacing variable v by y/x.

26.4.1 Example

Solve the differential equation

$$\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$$

Solution: Since the right hand side of the given equation is function of y/x alone, the given problem is homogeneous equation. Substituting $y/x = v$ so that

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \tag{26.3}$$

the given equation becomes

$$v + x\frac{dv}{dx} = v + \tan v \quad \rightarrow \quad \frac{dx}{x} = \frac{\cos v}{\sin v}dv$$

Integrating and substituting the value of v, we get the solution as

$$cx = \sin\left(\frac{y}{x}\right),$$

where c is an arbitrary constant.

26.5 Equations Reducible to Homogeneous Form

Equation of the form

$$\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}, \qquad \frac{a}{a'} \neq \frac{b}{b'} \tag{26.4}$$

can be reduced to homogeneous form. The procedure is as follows:

Take

$$x = X + h \text{ and } y = Y + k$$

where X, Y are new variables and h, k are constants to be chosen so that the resulting equation in X, Y becomes homogeneous. From above we have $dx = dX$, and $dy = dY$, so that $dy/dx = dY/dX$. Now the given differential equation in new variables becomes

$$\frac{dX}{dY} = \frac{aX + bY + (ah + bk + c)}{a'X + b'Y + (a'h + b'k + c')} \tag{26.5}$$

In order to make (26.5) homogeneous, the constant h and k must satisfy the following algebraic equations

$$ah + bk + c = 0 \quad , \quad a'h + b'k + c' = 0 \tag{26.6}$$

Solving equations (26.6), we obtain

$$h = \frac{bc' - b'c}{ab' - a'b} \quad , \quad k = \frac{ca' - c'a}{ab' - a'b} \tag{26.7}$$

provided $ab' - a'b \neq 0$. Knowing h and k we have

$$X = x - h, \qquad Y = y - k. \tag{26.8}$$

The Equation (26.5) now reduces to

$$\frac{dY}{dX} = \frac{aX + b(Y/X)}{a' + b'(Y/X)} \tag{26.9}$$

which is a homogeneous equation in X and Y which can be solved by substituting $Y/X = v$. After getting solution in X and y, we remove X and Y using (26.8) and obtain solution in terms of x and y.

26.5.1 Example

Solve the differential equation

$$\frac{dy}{dx} = \frac{(x + y + 4)}{(x - y - 6)}$$

Solution: Let $x = X + h, \quad y = Y + k,$ so that $dy/dx = dY/dX$ and using this, the given differential equation reduces to

$$\frac{dy}{dx} = \frac{X + Y + (h + k + 4)}{X - Y + (h - k - 6)}. \tag{26.10}$$

Choose h and k such that $h + k + 4 = 0, \quad h - k - 6 = 0$, and by solving, we get $h = 1$ and $k = -5$. New variables becomes $X = x - 1$ and $Y = y + 5$. Using this into (26.10), we obtain

$$\frac{dY}{dX} = \frac{1 + Y/X}{1 + Y/X}. \tag{26.11}$$

Substituting

$$Y = Xv \text{ and } \frac{dY}{dX} = v + X\frac{dv}{dX}$$

the Equation (26.11) becomes

$$\frac{dX}{X} = \frac{1 - v}{1 + v^2}dv = \frac{dv}{1 + v^2}dv - \frac{vdv}{1 + v^2}. \tag{26.12}$$

Integrating the above equation, we get

$$\log X = \tan^{-1} v - (1/2)\log(1 + v^2) + (1/2)\log c$$

Further simplifications gives

$$2\log X + \log(1 + Y^2/X^2) - \log c = 2\tan^{-1}(Y/X), \quad \text{as} \quad v = Y/X$$

Thus, we get

$$X^2 + Y^2 = ce^{2\tan^{-1}(Y/X)};$$

Replacing X and Y as $X = x - 1$ and $Y = y + 5$ we obtain the general solution as

$$(x-1)^2 + (y+5)^2 = ce^{2\tan^{-1}((y+5)/(x-1))}.$$

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Lesson 27

Linear Differential Equation of First Order

In this lesson we shall learn linear differential equations of first order. Such equations are very often used in applications. Solution strategies of solving such equations will be discussed. Further a another special form of differential equation which can be reduced to linear differential equation of first order will be studied.

27.1 Linear Differential Equation

A first order differential equation is called linear if it can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{27.1}$$

where P and Q are constants or function of x only.

A method of solving (27.1) relies on multiplying the equation by a function called integrating function so that the left hand side of the differential equation can be brought under a common derivative. Suppose $R(x)$ is an integrating factor of the (27.1). Multiplying the (27.1) by $R(x)$, we obtain

$$R(x)\frac{dy}{dx} + P(x)R(x)y = Q(x)R(x) \tag{27.2}$$

Suppose, we wish that the L.H.S of (27.2) is the differential coefficient of some product. Clearly, the term $R(x)\frac{dy}{dx}$ can only be obtained by differentiating the product $R(x)y(x)$. In other words, we wish to have

$$R(x)\frac{dy}{dx} + P(x)R(x)y(x) = \frac{d}{dx}(R(x)y(x)). \tag{27.3}$$

This implies

$$R(x)\frac{dy}{dx} + P(x)R(x)y(x) = R(x)\frac{dy}{dx} + y(x)\frac{dR}{dx}.$$

On cancelling the first term on both the sides we obtain

$$P(x)R(x)y(x) = y(x)\frac{dR}{dx} \qquad \Rightarrow \quad \frac{dR}{R} = Rdx.$$

Integrating the above equation, we get $\log R = \int P dx$. Note that the constant of integration is not important here because the integrating factor will be used to multiplying both the sides of the differential equation and therefore it will be cancelled. Thus, an integrating factor (I.F.) of the differential Equation (27.1) is

$$R = e^{\int P dx} \tag{27.4}$$

The Equation (27.2) now reduces to

$$\frac{d}{dx}(Ry) = QR$$

By integrating above equation, we have

$$Ry = \int RQdx + c,$$

or

$$ye^{\int P dx} = \int Qe^{\int P dx} dx + c,$$

which is required solution of given differential equation. Here C is the constant of integration.

27.2 Example Problems

27.2.1 Problem 1

Solve $x \cos x \frac{dy}{dx} + y(x \sin x + \cos x) = 1, \quad 0 < x < \pi/2.$

Solution: We rewrite the given equation as

$$\frac{dy}{dx} + (\tan x + \frac{1}{x}) = \frac{\sec x}{x}.$$

An I.F. of the given differential equation is

$$e^{\int (\tan x + \frac{1}{x}) dx} = e^{\log x \sec x} = x \sec x.$$

Hence, the required solution is

$$yx \sec x = \int \sec^2 x dx + c,$$

or

$$yx \sec x = \tan x + c,$$

where, c is an arbitrary constant.

27.2.2 Problem 2

Solve $(1+x^2)\dfrac{dy}{dx} = x(1-y)$.

Solution: Rewriting the given differential equation in standard form

$$\frac{dy}{dx} + \frac{x}{1+x^2}y = \frac{x}{1+x^2}.$$

The I.F. is

$$I.F. = e^{\int \frac{x}{1+x^2}dx} = e^{\frac{1}{2}\ln(1+x^2)} = \sqrt{1+x^2}.$$

The solution is

$$y\sqrt{1+x^2} = \int \frac{x}{\sqrt{1+x^2}} + c \quad \Rightarrow \quad y = 1 + c\,(1+x^2)^{-1/2}$$

Here c is an arbitrary constant.

27.3 Equations Reducible to Linear Form

A equation of the form

$$f'(y)\frac{dy}{dx} + Pf(y) = Q, \tag{27.5}$$

can be reduced to linear form, by substituting $f(y) = v$ so that $f'(y)\frac{dy}{dx} = dv/dx$. The Equation (27.5) then becomes

$$dv/dx + Pv = Q, \tag{27.6}$$

which is linear in v and x and its solution can be obtained with the help of I.F. as before. Thus, we have an I.F.= $e^{\int pdx}$ and the solution is

$$ve^{\int pdx} = \int Qe^{\int Pdx}dx + c.$$

Finally, we replace v by $f(y)$ to obtain the required solution.

27.4 Example Problems

27.4.1 Problem 1

Solve $\dfrac{dy}{dx}\cos y + 2x\sin y = x.$

Solution: Substitution $\sin y = v$ which implies $\cos y \frac{dy}{dx} = \frac{dv}{dx}$ reduces the given differential equation to

$$\frac{dv}{dx} + 2xv = x.$$

This is a linear differential equation of first order and its I.F. is $e^{\int 2xdx} = e^{x^2}$. The solution of the equation in v is given by

$$ve^{x^2} = \int xe^{x^2}dx + c \quad \Rightarrow v = \frac{1}{2} + ce^{-x^2}.$$

Replacing v by $\sin y$ we get the required solution as

$$y = \sin^{-1}\left(\frac{1}{2} + ce^{-x^2}\right).$$

27.4.2 Problem 2

Solve $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$

Solution: Dividing the given differential equation by $\cos^2 y$, we obtain

$$\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3.$$

Putting $\tan y = v$ so that $\sec^2 y \frac{dy}{dx} = \frac{dv}{dx}$. Hence the above equation becomes

$$\frac{dv}{dx} + 2xv = x^3,$$

which is linear. Its I.F. is e^{x^2} and its solution is given as follows

$$ve^{x^2} = \int e^{x^2} x^3 dx + c,$$
$$ve^{x^2} = \frac{1}{2}(x^2 - 1)e^{x^2} + c.$$

Replacing v by $\tan y$ we obtain the required solution.

27.5 Bernoulli's Equation

An equation of the form

$$dy/dx + Py = Qy^n \tag{27.7}$$

where P and Q are constants or function of x only and n is constant except 0 and 1 is called Bernoulli differential equation. This equation can easily be solved by multiplying both sides by y^{-n} as

$$y^{-n}dy/dx + Py^{1-n} = Q \tag{27.8}$$

Setting $y^{1-n} = v$, so that $y^{-n}\frac{dy}{dx} = \frac{1}{(1-n)}\frac{dv}{dx}$, the Equation (27.8) becomes

$$dv/dx + P(1-n)v = Q(1-n),$$

which is linear in v and x. Its I.F. is $e^{\int P(1-n)dx}$ and hence the required solution is

$$y^{1-n}e^{\int P(1-n)dx} = \int Qe^{\int P(1-n)dx}dx + c,$$

where c is an arbitrary constant.

27.5.1 Example

Solve $x\dfrac{dy}{dx} + y = y^2 \ln x$.

Solution: Rewrite the given equation

$$y^{-2}\frac{dy}{dx} + \frac{1}{x}y^{-1} = -x^{-1}\ln x \tag{27.9}$$

Putting $y^{-1} = v$ so that $-y^{-2}\frac{dy}{dx} = \frac{dv}{dx}$. Then the Equation (27.9) gives

$$\frac{dv}{dx} - \frac{1}{x}v = x^{-1}\ln x. \tag{27.10}$$

The I.F. of the differential Equation (27.10) is $e^{-\int \frac{1}{x}dx} = \frac{1}{x}$, and hence the solution becomes

$$v\frac{1}{x} = -\int x^{-2}\log x dx + c$$

or by replacing v by y^{-1} we get

$$y^{-1} = 1 + \ln x + cx,$$

where c is an arbitrary constant.

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Lesson 28

Exact Differential Equation of First Order

This lesson provides an overview of exact differential equation. A necessary condition for a differential equation to be exact will be derived. Then different solution techniques will be discussed. Several examples to clarify the ideas will be supplemented.

28.1 Exact Differential Equation of First Order

If M and N are functions of x and y, the equation $Mdx + Ndy = 0$ is called exact when there exists a function $f(x, y)$ such that

$$d(f(x, y)) = Mdx + Ndy,$$

or equivalently

$$\frac{\partial f}{\partial y}dx + \frac{\partial f}{\partial x}dy = Mdx + Ndy.$$

28.1.1 Theorem

The necessary and sufficient condition for the differential equation

$$Mdx + Ndy = 0 \tag{28.1}$$

to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{28.2}$$

Proof: First we proof that the condition (28.2) is necessary. To prove we let the Equation (28.1) to be exact. Then, by definition, there exists $f(x, y)$ such that

$$\frac{\partial f}{\partial y}dx + \frac{\partial f}{\partial x}dy = Mdx + Ndy. \tag{28.3}$$

Equating coefficients of dx and dy in Equation (28.3), we get

$$M = \frac{\partial f}{\partial y}, \tag{28.4}$$

$$N = \frac{\partial f}{\partial x}. \tag{28.5}$$

To eliminate the unknown $f(x, y)$ from above equations, we assume that the 2nd order partial derivatives of f are continuous. We now differentiate (28.4) and (28.5) w.r.t. x and y respectively as

$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial y \partial x}$$

This implies

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Thus, if (28.1) is exact, M and N satisfy (28.2).

Now we show that the condition is sufficient. Suppose (28.2) holds and show that (28.1) is exact. For this we find a function $f(x, y)$ such that

$$d(f(x, y)) = Mdx + Ndy.$$

Let $g(x, y) = \int Mdx$ be the partial integral of M such that $\frac{\partial g}{\partial x} = M$. We first prove that $\left(N - \frac{\partial g}{\partial y}\right)$ is function of y only. This is clear because

$$\frac{\partial}{\partial x}\left(N - \frac{\partial g}{\partial y}\right) = \frac{\partial N}{\partial x} - \frac{\partial^2 g}{\partial x \partial y}$$

Assuming $\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial^2 g}{\partial y \partial x}$ and using Equation (28.2) we get

$$\begin{aligned}\frac{\partial}{\partial x}\left(N - \frac{\partial g}{\partial y}\right) =& \frac{\partial N}{\partial x} - \frac{\partial^2 g}{\partial y \partial x} \\ =& \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}(\frac{\partial g}{\partial x}) = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0.\end{aligned}$$

Take, $f(x, y) = g(x, y) + \int (N - \frac{\partial g}{\partial y})dy$. Hence taking total differentiation of this equation gives

$$\begin{aligned}df = dg + (N - \frac{\partial g}{\partial y})dy &= \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + Ndy - \frac{\partial g}{\partial y}dy, \\ &= (\frac{\partial g}{\partial x})dx + Ndy = Mdx + Ndy,\end{aligned}$$

Thus, if Equation (28.2) is satisfied, Equation (28.1) is an exact equation. ■

28.2 Example Problems

28.2.1 Problem 1

Solve $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$.

Solution: Comparing the given equation with $Mdx + Ndy = 0$, we have

$$M = (x^2 - 4xy - 2y^2), \qquad N = (y^2 - 4xy - 2x^2)$$

Therefore

$$\frac{\partial M}{\partial y} = -4x - 4y = \frac{\partial N}{\partial x}$$

Hence, the given equation is exact and hence there exists a function $f(x, y)$ such that

$$d(f(x, y)) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = Mdx + Ndy$$

which implies

$$\frac{\partial f}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial f}{\partial y} = N(x, y)$$

Integration of the first of above equations with respect to x gives

$$f = \frac{1}{3}x^3 - 2x^2y - 2y^2x + c_1(y)$$

where $c_1(y)$ is an arbitrary function of y only. Differentiating the above f with respect to y and using $\frac{\partial f}{\partial y} = N(x, y)$ we get

$$\frac{\partial f}{\partial y} = -2x^2 - 4xy + c_1'(y) = +y^2 - 4xy - 2x^2$$

This implies

$$c_1'(y) = y^2 \Rightarrow c_1(y) = \frac{y^3}{3} + c_2$$

Hence the solution is given by

$$f(x, y) = c_3 \Rightarrow x^3 - 6xy(x + y) + y^3 = c$$

Here c_2, c_3 and c are constants of integration.

28.2.2 Problem 2

Determine whether the differential equation $(x + \sin y)dx + (x\cos y - 2y)dy = 0$ *is exact and solve it.*

Solution: For given equation we have

$$M(x, y) = (x + \sin y) \quad \text{and} \quad N(x, y) = (x\cos y - 2y) \tag{28.6}$$

Now we check

$$\frac{\partial M}{\partial y} = \cos y = \frac{\partial N}{\partial x}$$

Hence the given differential equation is exact. For the solution we seek a function $f(x, y)$ so that

$$\frac{\partial f}{\partial x} = (x + \sin y) \quad \text{and} \quad \frac{\partial f}{\partial y} = (x\cos y - 2y)$$

From the first relation we get

$$f(x, y) = \frac{x^2}{2} + x\sin y + c_1(y)$$

Differentiating w.r.t. y and using the second relation of (28.6) we get

$$x\cos y + c_1'(y) = x\cos y - 2y \quad \Rightarrow \quad c_1'(y) = -2y \Rightarrow c_1(y) = -y^2 + c_2$$

Therefore, we have

$$f(x, y) = \frac{x^2}{2} + x\sin y - y^2 + c_2$$

Then the solution of the given differential equation

$$f(x, y) = c_3 \quad \Rightarrow \quad \frac{x^2}{2} + x\sin y - y^2 = c.$$

28.2.3 Problem 3

Solve the differential equation $(2y^2x - 2y^3)dx + (4y^3 - 6y^2x + 2yx^2)dy$

Solution: First we check the exactness of the equation by

$$\frac{\partial M}{\partial y} = 4xy - 6y^2 = \frac{\partial N}{\partial x}$$

So the equation is exact. Then, there exists a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = (2y^2x - 2y^3) \quad \text{and} \quad \frac{\partial f}{\partial y} = (4y^3 - 6y^2x + 2yx^2)$$

This gives

$$f(x, y) = (y^2x^2 - 2xy^3) + c_1(y) \quad \Rightarrow \quad \frac{\partial f}{\partial y} = (2yx^2 - 6xy^2) + c_1'(y)$$

This implies

$$c_1'(y) = 4y^3 \quad \Rightarrow \quad c_1(y) = y^4 + c_2$$

Hence the solution is

$$f(x, y) = c_3 \quad \Rightarrow \quad y^2x^2 - 2xy^3 + y^4 = c.$$

28.2.4 Problem 4

Solve that the differential equation $(3xy + y^2)dx + (x^2 + xy)dy = 0$. *is not exact and hence it cannot be solve by the method discussed above.*

Solution: For the given differential equation we have

$$\frac{\partial M}{\partial y} = 3x + 2y, \quad \textit{and} \quad \frac{\partial N}{\partial x} = 2x + y;$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is not exact.

Now we see that it cannot be solved by the procedure described previously where we seek a function f such that

$$\frac{\partial f}{\partial x} = 3xy + y^2 \quad \text{and} \quad \frac{\partial f}{\partial y} = x^2 + xy \tag{28.7}$$

Integration of the first relation gives

$$f(x, y) = \frac{3}{2}x^2y + xy^2 + c_1(y)$$

where $c_1(y)$ is an arbitrary function of y only. Now we differentiate the above equation with respect to y and set the resulting expression equals to $x^2 + xy$ from the second relation of (28.7) as

$$\frac{3}{2}x^2 + 2xy + c_1'(y) = x^2 + xy$$

This provides

$$c_1'(y) = -\frac{1}{2}x^2 - xy$$

Since the right side of the above depends on x as well as on y, it is impossible to solve this equation for $c_1(y)$. Thus there is no $f(x, y)$ exists and hence the given differential equation cannot be solved in this way.

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Lesson 29

Exact Differential Equations: Integrating Factors

In general, equations of the type $M(x,y)dx + N(x,y)dy = 0$ are not exact. However, it is sometimes possible to transform the equation into an exact differential equation multiplying it by a suitable function $I(x,y)$. That is, if $I(x,y)$ is an integrating factor then the differential equation

$$I(x,y)M(x,y)dx + I(x,y)N(x,y)dy = 0$$

becomes exact. A solution to the above equation is obtained by solving the exact differential equation as in the previous lesson. Note that the given equation may have several integrating factors. This is exactly the procedure we have used for solving linear differential equations in earlier lesson. Here we deal with more general differential equation.

29.1 Rule I: By Inspection

There is not much theory behind finding integrating factor by inspection. This method works based on recognition of some standard exact differentials that occur frequently in practice. The following list of exact differentials would be quite useful in solving exact differential equations:

(i) $d(xy) = ydx + xdy$

(ii) $d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$ or $d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$

(iii) $d\left(\ln\frac{y}{x}\right) = \frac{xdy - ydx}{xy}$ or $d\left(\ln\frac{x}{y}\right) = \frac{ydx - xdy}{xy}$

(iv) $d\left(\arctan\frac{y}{x}\right) = \frac{xdy - ydx}{x^2 + y^2}$ or $d\left(\arctan\frac{x}{y}\right) = \frac{ydx - xdy}{y^2 + x^2}$

(v) $d\left(\ln xy\right) = \frac{ydx + xdy}{xy}$

29.1.1 Example

Solve the differential equation $y(y^2+1)dx + x(y^2-1)dy$.

Solution: The given equation can be rewritten as

$$y^2(ydx + xdy) + ydx - xdy$$

This is further rewritten as

$$(ydx + xdy) + \left(\frac{ydx - xdy}{y^2}\right) = 0$$

Using standard differential forms given above we get

$$d(xy) + d\left(\frac{x}{y}\right) = 0$$

Integrating the above equation, the desired solution is given as

$$xy^2 + x = cy$$

Here c is an arbitrary constant.

29.2 Rule II: $Mdx + Ndy = 0$ is homogeneous and $Mx + Ny \neq 0$

If the equation $Mdx+Ndy = 0$ is homogeneous and $Mx+Ny \neq 0$, then $I(x, y) = \dfrac{1}{(Mx + Ny)}$ is an integrating factor. In order to prove the result, we need to show that

$$\frac{Mdx + Ndy}{Mx + Ny} = d\,(\text{some function } x \text{ and } y)$$

Rewriting $Mdx + Ndy$ as

$$Mdx + Ndy = \frac{1}{2}\left\{(Mx + Ny)\left(\frac{dx}{x} + \frac{dy}{y}\right) + (Mx - Ny)\left(\frac{dx}{x} - \frac{dy}{y}\right)\right\}$$

Multiplying by proposed integrating factor we get

$$\frac{Mdx + Ndy}{Mx + Ny} = \frac{1}{2}\left\{\left(\frac{dx}{x} + \frac{dy}{y}\right) + \frac{(Mx - Ny)}{(Mx + Ny)}\left(\frac{dx}{x} - \frac{dy}{y}\right)\right\} \qquad (29.1)$$

Given that $M(x, y)$ and $N(x, y)$ are homogeneous functions of some degree n, i.e., $M(tx, ty) = t^n M(x, y)$ and $N(x, y) = t^n N(x, y)$. Then

$$M\left(\frac{x}{y}, 1\right) = M\left(\frac{1}{y}x, \frac{1}{y}y\right) = \frac{1}{y^n}M(x, y) \quad \Rightarrow \quad M(x, y) = y^n M\left(\frac{x}{y}, 1\right)$$

Similarly, we get

$$N(x,y) = y^n N\left(\frac{x}{y}, 1\right)$$

Now consider

$$\frac{(Mx - Ny)}{(Mx + Ny)} = \frac{y^n x M\left(\frac{x}{y}, 1\right) - y^n y N\left(\frac{x}{y}, 1\right)}{y^n x M\left(\frac{x}{y}, 1\right) + y^n y N\left(\frac{x}{y}, 1\right)} = \frac{\frac{x}{y} M\left(\frac{x}{y}, 1\right) - N\left(\frac{x}{y}, 1\right)}{\frac{x}{y} M\left(\frac{x}{y}, 1\right) + N\left(\frac{x}{y}, 1\right)} = f\left(\frac{x}{y}\right)$$

Going back to the Equation (29.1), we have

$$\frac{Mdx + Ndy}{Mx + Ny} = \frac{1}{2}\left\{d\left(\ln(xy)\right) + f\left(\frac{x}{y}\right) d\left(\ln\frac{x}{y}\right)\right\}$$

Rewriting $f\left(x/y\right) = f\left(\exp(\ln(x/y))\right)$ and defining $g(x) := f(\exp(x))$, the above equation becomes

$$\frac{Mdx + Ndy}{Mx + Ny} = \frac{1}{2}\left\{d\left(\ln(xy)\right) + g\left(\ln(x/y)\right) d\left(\ln\frac{x}{y}\right)\right\}$$

Hence, we have shown that

$$\frac{Mdx + Ndy}{Mx + Ny} = d\left[\frac{1}{2}\ln(xy) + \frac{1}{2}\int g\left(\ln\frac{x}{y}\right) d\left(\ln\frac{x}{y}\right)\right]$$

Thus $\frac{1}{Mx + Ny}$ is an integrating factor of the homogenous differential equation $Mdx + Ndy = 0$.

29.2.1 Example

Solve the differential equation $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$

Solution: The given equation is a homogeneous differential equation. Comparing it with $Mdx + Ndy = 0$, we have $M = x^2y - 2xy^2$ and $N = -(x^3 - 3x^2y)$. Since

$$Mx + Ny = (x^2y - 2xy^2)x - y(x^3 - 3x^2y) = x^2y^2 \neq 0,$$

the integrating factor is

$$\frac{1}{(Mx + Ny)} = \frac{1}{x^2y^2}$$

Multiply by the integrating factor, the given differential equation becomes

$$(1/y - 2/x)dx - (x/y^2 - 3/y)dy = 0$$

This is now exact and can be rewritten as

$$\frac{ydx - xdy}{y^2} - \frac{2}{x}dx + \frac{3}{y}dy = 0 \quad \Rightarrow \quad d\left(\frac{x}{y}\right) - \frac{2}{x}dx + \frac{3}{y}dy = 0$$

Integrating the above equation we obtain the desired solution as

$$x - 2y\ln x + 3y\ln y = cy$$

29.3 Rule III: $Mdx + Ndy = 0$ is of the form $f_1(xy)ydx + f_2(xy)xdy = 0$

If the equation $Mdx + Ndy = 0$ is of the form $f_1(xy)ydx + f_2(xy)xdy = 0$, then $\frac{1}{(Mx - Ny)}$ is an integrating factor provided $Mx - Ny \neq 0$. Similar to rule II we now show that

$$\frac{Mdx + Ndy}{Mx - Ny} = d\,(\text{some function } x \text{ and } y)$$

Again, rewriting $Mdx + Ndy$ as

$$Mdx + Ndy = \frac{1}{2}\left\{(Mx + Ny)\left(\frac{dx}{x} + \frac{dy}{y}\right) + (Mx - Ny)\left(\frac{dx}{x} - \frac{dy}{y}\right)\right\}$$

Now dividing by $Mx - Ny$ we get

$$\frac{Mdx + Ndy}{Mx - Ny} = \frac{1}{2}\left\{\frac{(Mx + Ny)}{Mx - Ny}\left(\frac{dx}{x} + \frac{dy}{y}\right) + \left(\frac{dx}{x} - \frac{dy}{y}\right)\right\}$$

Using $M = f_1(xy)y$ and $N = f_2(xy)x$ we obtain

$$\frac{Mdx + Ndy}{Mx - Ny} = \frac{1}{2}\left\{\frac{f_1(xy) + f_2(xy)}{f_1(xy) - f_2(xy)}d\,(\ln xy) + d\left(\ln\frac{x}{y}\right)\right\}$$

Let $f(xy) := \frac{f_1(xy) + f_2(xy)}{f_1(xy) - f_2(xy)}$ and $g(x) := f(\exp(x))$, the above equation reduces to

$$\frac{Mdx + Ndy}{Mx - Ny} = \frac{1}{2}\left\{f(xy)d\,(\ln xy) + d\left(\ln\frac{x}{y}\right)\right\} = \frac{1}{2}\left\{g(\ln xy)d\,(\ln xy) + d\left(\ln\frac{x}{y}\right)\right\}$$

This shows that

$$\frac{Mdx + Ndy}{Mx - Ny} = d\left[\frac{1}{2}\int g(\ln xy)d\,(\ln xy) + \frac{1}{2}\left(\ln\frac{x}{y}\right)\right]$$

29.3.1 Example

Solve $y(x^2y^2+2)dx + x(2-2x^2y^2)dy = 0$.

Solution: Comparing with $Mdx+Ndy = 0$, we have $M = y(x^2y^2+2)$ and $N = x(2-2x^2y^2)$. The given equation is of the form

$$f_1(xy)ydx + f_2(xy)xdy = 0$$

and we have

$$Mx - Ny = xy(x^2y^2+2) - xy(2-2x^2y^2) = 3x^3y^3 \neq 0$$

Therefore, multiplying the equation by $1/3x^3y^3$, we obtain

$$(1/3x + 2/(3x^3y^2))dx + (2/(3x^2y^3) - 2/3y)dy = 0$$

This is an exact differential equation which can be solved with the technique discussed in previous lesson.

29.4 Rule IV: Most general approach

Now we discuss the most general approach of finding integrating function. The idea is to multiply the given differential equation

$$M(x,y)dx + N(x,y)dy = 0 \tag{29.2}$$

by a function $I(x,y)$ and then try to choose $I(x,y)$ so that the resulting equation

$$I(x,y)M(x,y)dx + I(x,y)N(x,y)dy = 0 \tag{29.3}$$

becomes exact. The above equation is exact if and only if

$$\frac{\partial(IM)}{\partial y} = \frac{\partial(IN)}{\partial x} \tag{29.4}$$

If a function $I(x,y)$ satisfying the partial differential Equation (29.4) can be found, then (29.3) will be exact. Unfortunately, solving Equation (29.4), is as difficult to solve as the original Equation (29.2) by some other methods. Therefore, while in principle integrating factors are powerful tools for solving differential equations, in practice they can be found

only in special cases. The cases we will consider are: (i) an integrating factor I that is either as function of x only, or (ii) a function of y only.

Let us determine necessary conditions on M and N so that (29.2) has an integrating factor I that depends on x only. Assuming that I is a function of x only, then Equation (29.4) reduces to

$$IM_y = IN_x + N\frac{dI}{dx} \quad \Rightarrow \quad \frac{dI}{dx} = \frac{IM_y - IN_x}{N} \tag{29.5}$$

If $(M_y - N_x)/N$ is a function of x only, say f(x), then there is an integrating factor I that also depends only on x which can be found by solving (29.5) as $I(x) = e^{\int f(x)dx}$. A similar procedure can be used to determine a condition under which Equation (29.2) has an integrating factor depending only on y. To conclude, we have:

If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is function of x alone say $f(x)$, then $I(x) = e^{\int f(x)dx}$ is an I.F.
If $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ is function of y alone say $f(y)$, then $I(y) = e^{\int f(y)dy}$ is an I.F.

29.5 Example Problems

29.5.1 Problem 1

Find an integrating factor of $(x^2 + y^2 + x)dx + xydy = 0$ **Solution:** Comparing with $Mdx + Ndy = 0$, we have

$$M = (x^2 + y^2 + x) \text{ and } N = xy$$

Further, note that

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{x}$$

is a function of x alone. Hence, the integrating factor of the given problem is $e^{\int 1/xdx} = x$.

29.5.2 Problem 2

Find an integrating factor of $(2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0$

Solution: Compare with $Mdx + Ndy = 0$, we get

$$M = (2xy^4e^y + 2xy^3 + y) \text{ and } N = (x^2y^4e^y - x^2y^2 - 3x)$$

Also, note that

$$\frac{1}{M}\left(\frac{\partial N}{\partial y}-\frac{\partial M}{\partial x}\right)=-\frac{4}{y}$$

is a function of y alone. Hence the integrating factor of the given problem is $e^{\int -4/ydy}=1/y^4$.

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Lesson 30

Linear Differential Equations of Higher Order

In this lesson we discuss linear differential equation of higher order with constant coefficients. In particular, we shall learn about the techniques of finding solutions of homogenous equations. Different cases will be considered with the help of several examples.

30.1 Linear Differential Equation

In a linear differential equation, the dependent variable and its differential coefficients occur only in the first degree and are not multiplied together. The general form of the equation is

$$\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + a_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_n(x)y = F(x), \tag{30.1}$$

where $a_1, a_2, \ldots, a_n$ and F are either constants or functions of x only. If the right hand side, i.e. $F(x)$, is identically zero, the equation is said to be homogeneous; otherwise it is called nonhomogeneous. Before we discuss some particular cases of the above equation we state two facts about the solution of a linear homogeneous differential equation. The first says that if we know n solutions $y_1, y_2, \ldots, y_n$ of the linear homogeneous equation, then any linear combination $y = c_1y_1 + c_2y_2 + \ldots c_ny_n$ is also a solution for any constants $c_1, c_2, \ldots, c_n$. This can easily be proved by substituting $y = c_1y_1 + c_2y_2 + \ldots c_ny_n$ into the equation and using linearity of the equation. The second important result concerns about the general solutions (solution containing all solutions) to the linear homogeneous equation. This result says that any solution is some linear combination of $y_1, y_2, \ldots, y_n$ for some suitable values of constants $c_1, c_2, \ldots, c_n$. However, this is not true for any combination of solutions but is true if the solutions $y_1, y_2, \ldots, y_n$ are linearly independent.

30.2 Linear Differential Equation with Constant Coefficients

An equation of the form

$$\frac{d^n y}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_ny = F(x), \tag{30.2}$$

where $a_1, a_2, \ldots, a_n$ are constants, is called linear differential equation with constant coefficients. Using the symbols $D^n := \dfrac{d^n}{dx^n}$, the Equation (30.2) becomes

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + ... + a_n)y = F(x), \tag{30.3}$$

Further defining $f(D) := D^n + a_1 D^{n-1} + a_2 D^{n-2} + ... + a_n$, we can rewrite the given differential equation in a more compact form as $f(D)y = F(x)$. Here $f(D)$ acts as operator on y to yield $F(x)$. The general solution of (30.2) can be written as the sum of the general solution of the corresponding homogeneous equation, refereed as complimentary function (C.F.), and a particular solution or sometimes called particular integral (P.I) of nonhomogeneous equation. Thus

$$y = C.F. + P.I. \tag{30.4}$$

Note that the C.F. involves n arbitrary constants and P.I. does not involve any arbitrary constant. It is readily evident that y in (30.4) is the general solution of the given nonhomogeneous differential equation because it satisfies the given differential equation as $f(D)\,(C.F. + P.I.) = f(D)\,(C.F.) + f(D)\,(P.I.) = 0 + F(x)$ and it has n arbitrary constants.

30.3 C.F. of a Differential Equation

By definition, C.F. of (30.2) is the general solution of

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + ... + a_n)y = 0 \tag{30.5}$$

To solve Equation (30.5), we seek a function which satisfies the above equation. One intelligent guess of such a function is the exponential function e^{mx}, where m is a constant. Differentiations of this exponential function are just constant multiples of the original exponential. If we substitute this function into the Equation (30.5), we obtain

$$(m^n + a_1 m^{n-1} + a_2 m^{n-2} + ... + a_n)e^{mx} = 0 \tag{30.6}$$

Since the exponential function is never zero, we can divide this last equation by e^{mx}. Thus, $y = e^{mx}$ is a solution to Equation (30.5) if and only if m is a solution to the algebraic equation

$$m^n + a_1 m^{n-1} + a_2 m^{n-2} + ... + a_n = 0 \tag{30.7}$$

Equation (30.7) is called the auxiliary equation (A.E.) or characteristic equation (C.E.) of the differential Equation (30.5).

30.4 Case I: A.E. has real and distinct roots

If $m_1, m_2, m_3, ..., m_n$ be real and distinct then the solutions $e^{m_1 x}, e^{m_m x}, \ldots, e^{m_n x}$ are linearly independent and the general solution of the given homogeneous differential equation becomes

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + ... + c_n e^{m_n x},$$

where $c_1, c_2, ..., c_n$ are arbitrary constants.

30.4.1 Example

Find the general solution of the differential equation $(D^3 + 6D^2 + 11D + 6)y = 0$.

Solution: The A.E. is $(m^3 + 6m^2 + 11m + 6) = 0$. The roots are $m = -1, -2, -3$. Hence the required solution is $y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$.

30.5 Case II: A.E. has repeated real roots

Let $m_1 = m_2$ are repeated roots of the A.E. Then, we have $n - 1$ linearly independent solutions. It can be shown that a simple choice $y = xe^{m_1 x}$ is also a solution which is independent to the rest $n - 1$ solutions. Thus, the general solution of the given differential equation is given by

$$y = (c_1 + c_2 x)e^{m_1 x} + c_3 e^{m_3 x} + ... + c_n e^{m_n x}$$

The above idea can be further extended by taking solutions $xe^{m_1 x}, x^2 e^{m_1 x}, \ldots, x^{l-1} e^{m_1 x} \ldots$ if the root m_1 is repeating $l-$times.

30.5.1 Example

Find the general solution to $(D^4 + 2D^3 - 3D^2 - 4D + 4)y = 0$.

Solution: The A.E. of given equation is

$$(m^4 + 2m^3 - 3m^2 - 4m + 4) = 0$$

The roots of the A.E. are $m = 1, 1, -2, -2$. The required solution is $y = (c_1 + c_2 x)e^x + (c_3 + c_4 x)e^{-2x}$.

30.6 Case III: A.E. has complex roots

If $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, then the solutions $e^{m_1 x}, e^{m_2 x}, \ldots, e^{m_n x}$ are linearly independent and the general solution of the given homogeneous differential equation is given by

$$y = c_1' e^{m_1 x} + c_2' e^{m_2 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}.$$

The above solution can be simplified as

$$y = c_1' e^{\alpha x} (\cos \beta x + i \sin \beta x) + c_2' e^{\alpha x} (\cos \beta x - i \sin \beta x) + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}.$$

Defining new constants $c_1 = c_1' + c_2'$ and $c_2 = i(c_1 - c_2)$, the general solution becomes

$$y = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}.$$

Similar to the case II, the solution for repeated complex roots can be found, see example below.

30.6.1 Example

Find the general solution to the differential equation $(D^2 + 1)^2 y = 0$.

Solution: The A.E. and its roots are

$$(m^2 + 1)^2 = 0, \quad \text{and therefore} \quad m = \pm i, \pm i.$$

This is the case of repeated complex root, so case II and case III can be combined to give the desired solution as $y = (c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x$.

30.7 Miscellaneous Problems

30.7.1 Problem 1

Find the general solution of the differential equation $(D^3 + 3D^2 + 3D + 1)y = 0$.

Solution: The A.E. and its root are given by $(m+1)^3 = 0$ and $m = -1, -1, -1$. Therefore, the required solution is $y = (c_1 + c_2 x + c_3 x^2) e^{-x}$.

30.7.2 Problem 2

Find the general solution of $(D^3 - 8)y = 0$.

Solution: The A.E. of the given equation is $(m^3 - 8) = 0$. Its root are $m = 2, -1 \pm i\sqrt{3}$. The required solution is $y = c_1 e^{2x} + e^{-x}(c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x)$.

30.7.3 Problem 3

Find the general solution of the differential equation $(D^2 - 2D + 5)^2 y = 0$.

Solution: The auxiliary equation is $(m^2 - 2m + 5)^2 = 0$. Its roots are $m = 1 \pm 2i, 1 \pm 2i$ Hence the required solution is $y = e^x \left[(c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x\right]$.

30.7.4 Problem 4

Find the general solution of $(D^2 + D + 1)^2(D - 2)y = 0$.

Solution: The A.E. of the given equation is $(m^2 + m + 1)^2(m - 2) = 0$. Its roots are $m = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}, 2$. Hence, the desired solution is

$$y = c_1 e^{2x} + e^{-\frac{1}{2}x}\left[(c_2 + c_3 x)\cos\frac{\sqrt{3}}{2}x + (c_4 + c_5 x)\sin\frac{\sqrt{3}}{2}x\right]$$

30.7.5 Problem 5

Find the general solution of the differential equation $(D^2 + 1)^3(D^2 + D + 1)^2 y = 0$.

Solution: The A.E. of given equation is

$$(D^2 + 1)^3(D^2 + D + 1)^2 = 0,$$

The roots are $m = \pm i, \pm i, \pm i, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. Therefore, the desired solution is

$$\begin{aligned} y =& (c_1 + c_2 x + c_3 x^2)\cos x + (c_4 + c_5 x + c_6 x^2)\sin x \\ &+ e^{-\frac{1}{2}x}\left[(c_7 + c_8 x)\cos\frac{\sqrt{3}}{2}x + (c_9 + c_{10} x)\sin\frac{\sqrt{3}}{2}x\right] \end{aligned}$$

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Lesson 31

Linear Differential Equation of Higher Order

In connection to the last lesson, we discuss solution methodologies of getting particular integral of the linear differential equations of higher order. In particular, in this lesson we present operator method which is somewhat easier than other methods for finding particular integrals.

31.1 Determination of Particular Integral (P.I.)

As we have seen in the earlier lesson that a general nonhomogeneous linear differential equations with constant coefficients can be written in operator form as $f(D)y = F(x)$. The operator, $1/f(D)$ is called inverse operator which gives a particular integral when operated on both the sides of the given differential equation. Hence, a particular integral of the given differential equation is given as $\frac{1}{f(D)}F(x)$. First we give a rather general idea of getting a particular integral with this method and then state some other useful direct results. Note that the operator $f(D)$ can be expressed as $(D-\alpha_1)(D-\alpha_2)\ldots(D-\alpha_n)$ and thus a particular integral is given as

$$\frac{1}{f(D)}F(x) = \frac{1}{D-\alpha_1}\frac{1}{D-\alpha_2}\ldots\frac{1}{D-\alpha_n}F(x) \tag{31.1}$$

We give a general idea of evaluating an expression of the type $\dfrac{1}{D-\alpha}F(x)$. This procedure can be repeatedly applied to find a particular integral (31.1). However, applicability of this method depends upon the form of $F(x)$.

We give a general theorem that can be applied to any problem for finding particular integral of a differential equation.

31.1.1 Theorem 1

If $F(x)$ is function of x and α is a constant, then

$$\frac{1}{D-\alpha}F(x) = e^{\alpha x}\int F(x)e^{-\alpha x}dx.$$

Proof: Let us assume that

$$y = \frac{1}{D-\alpha}F(x)$$

On operating $(D-\alpha)$ both sides, we get

$$(D-\alpha)y = F(x) \quad \Rightarrow \quad \frac{dy}{dx} - \alpha y = F(x)$$

The above equation is a linear differential equation of first order whose integrating factor is $e^{-\int \alpha dx} = e^{-\alpha x}$. Hence, the solution is given by

$$ye^{-\alpha x} = \int F(x)e^{-\alpha x}dx \quad \Rightarrow \quad y = e^{\alpha x}\int F(x)e^{-\alpha x}dx$$

Since our interest is finding a particular integrals, the constant of integration is dropped. Thus,

$$\frac{1}{D-\alpha}F(x) = e^{\alpha x}\int F(x)e^{-\alpha x}dx.$$ ■

Now we state some useful result those will be used to find P.I. of certain special forms of $F(x)$.

31.1.2 Theorem 2

If α is a constant, then $f(D)e^{\alpha x} = f(\alpha)e^{\alpha x}$

Proof: We know that $De^{\alpha x} = \alpha e^{\alpha x}$ and similarly $D^2e^{\alpha x} = \alpha^2 e^{\alpha x}$. With induction we can prove that $D^n e^{\alpha x} = \alpha^n e^{\alpha x}$ for any natural number n. This proves the result $f(D)e^{\alpha x} = f(\alpha)e^{\alpha x}$. ■

31.1.3 Theorem 3

If α is a constant and $g(x)$ is any function, then $f(D)\left(e^{\alpha x}g(x)\right) = e^{\alpha x}f(D+\alpha)g(x)$

Proof: We know that $D\left(e^{\alpha x}g(x)\right) = \alpha e^{\alpha x}g(x) + e^{\alpha x}Dg(x) = e^{\alpha x}(\alpha + D)g(x)$. Similar to the proof of previous theorem we can prove with induction that $D^n e^{\alpha x}g(x) = e^{\alpha x}(\alpha + D)^n g(x)$ for any natural number n. This proves the result $f(D)\left(e^{\alpha x}g(x)\right) = e^{\alpha x}f(D+\alpha)g(x)$. This result is known as shifting property of operator $f(D)$. ■

31.1.4 Theorem 4

If α and β are arbitrary constants, then

$$f(D^2)\sin(\alpha x+\beta) = f(-\alpha^2)\sin(\alpha x+\beta) \quad \textit{and} \quad f(D^2)\cos(\alpha x+\beta) = f(-\alpha^2)\cos(\alpha x+\beta)$$

Proof: It can easily be verified that $D^2\sin(\alpha x+\beta) = -\alpha^2\sin(\alpha x+\beta)$ and $D^2\cos(\alpha x+\beta) = -\alpha^2\cos(\alpha x+\beta)$. In other words, we can replace D^2 by $-\alpha^2$ and this proves the desired result. ■

Now we describe the method for some special form of $F(x)$.

31.2 Rule I: $F(x)$ is of the form e^{ax}

We know from Theorem 31.1.2 that $f(D)e^{\alpha x} = f(\alpha)e^{\alpha x}$. Operating on both sides by $1/f(D)$ we get

$$e^{\alpha x} = \frac{1}{f(D)}f(\alpha)e^{\alpha x} \quad \Rightarrow \quad e^{\alpha x} = f(\alpha)\frac{1}{f(D)}e^{\alpha x}$$

This implies that

$$\frac{1}{f(D)}e^{\alpha x} = \frac{1}{f(\alpha)}e^{\alpha x}, \quad \text{provided} \quad f(\alpha)\neq 0$$

If $f(\alpha)=0$, then $(D-\alpha)$ is a factor of $f(D)$, say $f(D)=(D-\alpha)g(D)$. Then

$$\frac{1}{f(D)}e^{\alpha x} = \frac{1}{(D-\alpha)}\frac{1}{g(D)}e^{\alpha x} = \frac{1}{(D-\alpha)}\frac{1}{g(\alpha)}e^{\alpha x} \quad \text{provided} \quad g(\alpha)\neq 0$$

Now using Theorem 31.1.1, we get

$$\frac{1}{f(D)}e^{\alpha x} = \frac{1}{g(\alpha)}\frac{1}{(D-\alpha)}e^{\alpha x} = \frac{1}{g(\alpha)}e^{\alpha x}x$$

In case $g(\alpha)=0$ then , say $f(D)=(D-\alpha)^2h(D)$. In this case we get

$$\frac{1}{f(D)}e^{\alpha x} = \frac{1}{h(\alpha)}\frac{1}{(D-\alpha)^2}e^{\alpha x} = \frac{1}{g(\alpha)}\frac{x^2}{2!}e^{\alpha x} \quad \text{provided} \quad h(\alpha)\neq 0$$

Again, if $h(\alpha)=0$, the same procedure can be repeated. To conclude, we have the following results:

(i) $\frac{1}{f(D)}e^{\alpha x} = \frac{1}{f(\alpha)}e^{\alpha x}$, where $f(\alpha)\neq 0$

(ii) If $f(\alpha) = 0$, then $f(D)$ must posses a factor of the type $(D-\alpha)^r$, say $f(D) = (D-\alpha)^r g(D)$ where $g(\alpha) \neq 0$. Then the following formula is applicable

$$\frac{1}{(D-\alpha)^r} e^{\alpha x} = \frac{x^r}{r!} e^{\alpha x}$$

.

31.3 Example Problems

31.3.1 Problem 1

Find the general solution of the differential equation $(D^2 - 3D + 2)y = e^{3x}$.

Solution: The auxiliary equation is

$$(m^2 - 3m + 2) = 0 \quad \Rightarrow \quad (m-1)(m-2) = 0 \quad \Rightarrow \quad m = 1, 2.$$

The complimentary function is given as

$$\text{C.F.} = c_1 e^x + c_2 e^{2x}$$

The particular integral is

$$\text{P.I.} = \frac{1}{D^2 - 3D + 2} e^{3x} = \frac{1}{3^2 - 3.3 + 2} e^{3x} = \frac{1}{2} e^{3x}.$$

The general solution is: $y = c_1 e^x + c_2 e^{2x} + \frac{1}{2} e^{3x}$.

31.3.2 Problem 2

Solve $(4D^2 - 12D + 9)y = 144 e^{3x/2}$

Solution: The auxiliary equation is

$$(4m^2 - 12m + 9) = 0 \quad \Rightarrow \quad m = 3/2, 3/2.$$

The complimentary function is

$$\text{C.F.} = (c_1 + c_2 x) e^{3x/2}$$

The particular integral is

$$\text{P.I.} = \frac{144}{(2D-3)^2} e^{3x/2} = \frac{144}{4} \frac{1}{(D-3/2)^2} e^{3x/2} = 36 \frac{x^2}{2!} e^{3x/2}$$

The required solution is: $y = (c_1 + c_2 x) e^{3x/2} + 36 \frac{x^2}{2!} e^{3x/2}$.

31.4 Rule II: $F(x)$ is of the form $\cos ax$ or $\sin ax$

We express $f(D)$ as a function of D^2, say $f(D) = \phi(D^2)$. From Theorem 31.1.4 we know that $\phi(D^2)\sin(\alpha x+\beta) = \phi(-\alpha^2)\sin(\alpha x+\beta)$. Applying $[\phi(D^2)]^{-1}$ both sides we obtain

$$\sin(\alpha x+\beta) = \frac{1}{\phi(D^2)}\phi(-\alpha^2)\sin(\alpha x+\beta)$$

If $\phi(-\alpha^2) \neq 0$, we can divide the above equation by $\phi(-\alpha^2)$ to get

$$\frac{1}{\phi(D^2)}\sin(\alpha x+\beta) = \frac{1}{\phi(-\alpha^2)}\sin(\alpha x+\beta)$$

Similarly,

$$\frac{1}{\phi(D^2)}\cos(\alpha x+\beta) = \frac{1}{\phi(-\alpha^2)}\cos(\alpha x+\beta), \quad \text{provided} \quad \phi(-\alpha^2)\neq 0$$

In case, $\phi(-\alpha^2) = 0$, we can rewrite $\sin(\alpha x+\beta) = \text{Im}(e^{i(\alpha x+\beta)})$ and $\cos(\alpha x+\beta) = \text{Re}(e^{i(\alpha x+\beta)})$. Now case I can be applied as

$$\frac{1}{f(D)}\sin(\alpha x+\beta) = \text{Im}\left(\frac{1}{f(D)}e^{i(\alpha x+\beta)}\right) = \text{Im}\left(\frac{1}{f(i\alpha)}e^{i(\alpha x+\beta)}\right) \quad \text{provided} \quad f(i\alpha)\neq 0$$

Similarly,

$$\frac{1}{f(D)}\cos(\alpha x+\beta) = \text{Re}\left(\frac{1}{f(i\alpha)}e^{i(\alpha x+\beta)}\right) \text{ provided } f(i\alpha)\neq 0$$

31.5 Example Problems

31.5.1 Problem 1

Solve the differential equation $(D^2+1)y = \cos 2x$.

Solution: The characteristic equation of the corresponding homogeneous equation is

$$(m^2+1) = 0 \quad \Rightarrow \quad m = \pm i$$

Hence, C.F. $= (c_1\cos x + c_2\sin x)$. The particular integral is given by

$$\text{P.I.} = \frac{1}{D^2+1}\cos 2x = \frac{1}{(-2^2+1)}\cos 2x = \frac{1}{-3}\cos 2x.$$

The required solution is: $y = (c_1\cos x + c_2\sin x) - \frac{1}{3}\cos 2x.$

31.5.2 Problem 2

Solve the differential equation $(D^2 - 4D + 3)y = \sin x$.

Solution: The roots of the characteristic equations are 1 and 3. The complementary function is $C.F. = c_1 e^x + c_2 e^{3x}$. The particular integral is

$$\text{P.I.} = \frac{1}{D^2 - 4D + 3} \sin x$$

Replacing D^2 by -1, we get

$$\text{P.I.} = \frac{1}{2 - 4D} \sin x = \frac{1}{2} \frac{1}{1 - 2D} \sin x = \frac{1}{2} \frac{1 + 2D}{1 - 4D^2} \sin x$$

Again, replacing D^2 by -1, we obtain

$$\text{P.I.} = \frac{1}{10}(1 + 2D) \sin x = \frac{1}{10}(\sin x + 2 \cos x)$$

Hence the complete solution is

$$y = c_1 e^x + c_2 e^{3x} + \frac{1}{10}(\sin x + 2 \cos x),$$

where c_1 and c_2 are arbitrary constants.

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Lesson 32

Linear Differential Equation of Higher Order (Cont.)

Here we continue discussion for solving linear equation of the form $f(D)y = F(x)$. In the last lesson, we have found particular integral for two different types of functions $F(x)$. In this lesson we shall continue discussing various other situations for finding particular integral.

32.1 Rule III: $F(x)$ is a polynomial of degree l

Take out the lowest degree term from $f(D)$, so as to reduce it in the form $[1 \pm f(D)]^n$. Take it to numerator, i.e., $[1 \pm f(D)]^{-n}$ and expand it in ascending powers of D with the help of Binomial series:

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + ...$$

Note that in the expansion we do not need to consider terms with power more than l, since $l+1$th and higher order derivatives of the polynomial of degree l will be zero.

32.1.1 Example

Solve the differential equation $(D^2 + D)y = x^2 + 2x + 4$

Solution: The characteristic equation of the corresponding homogeneous equation is

$$(m^2 + m) = 0 \quad \Rightarrow \quad m = 0, -1.$$

The complementary function is $c_1 + c_2 e^{-x}$. The particular integral is

$$\text{P.I.} = \frac{1}{D^2 + D}x^2 + 2x + 4 = \frac{1}{D}\frac{1}{(1+D)}x^2 + 2x + 4$$

Taking $1 + D$ into numerator and expending this into an infinite series we get

$$\text{P.I.} = \frac{1}{D}(1 - D + D^2 - D^3 + ...)(x^2 + 2x + 4) = \frac{1}{D}(x^2 + 2x + 4 - 2x - 2 + 2)$$

Operating $1/D$ on each term, we obtain P.I. $= \frac{1}{D}(x^2+4) = (x^3/3 + 4x)$. The desired general solution is

$$y = c_1 + c_2 e^{-x} + \left(\frac{x^3}{3} + 4x\right).$$

32.2 Rule IV: $F(x)$ is of the form $e^{\alpha x}V$, where V is any function of x

Using shift property of the operator discussed in the last lesson we can easily prove that

$$\frac{1}{f(D)}e^{\alpha x}V = e^{\alpha x}\frac{1}{f(D+\alpha)}V.$$

32.2.1 Example

Solve $(D^2 - 2D + 1)y = x^2e^x$.

Solution: The characteristic equation and its roots are

$$m^2 - 2m + 1 = 0, \quad \text{and} \quad m = 1, 1.$$

Thus, the complimentary function is

$$\text{C.F.} = (c_1 + c_2x)e^x$$

The particular integral is

$$\text{P.I.} = \frac{1}{D^2 - 2D + 1}x^2e^x = \frac{1}{(D-1)^2}x^2e^x$$

Using shift property we get

$$\begin{aligned}\text{P.I.} &= e^x\frac{1}{(D-1+1)^2}x^2,\\ &= e^x\frac{1}{D^2}x^2 = e^x\frac{1}{D}\left(\frac{1}{D}x^2\right)\\ &= e^x\frac{1}{D}\left(\frac{x^3}{3}\right) = e^x\frac{x^4}{12}.\end{aligned}$$

The required solution is $y = (c_1 + c_2x)e^x + e^x\frac{x^4}{12}$.

32.3 Rule V: $F(x)$ is of the form xV, where V is any function of x

Here, we prove the following result

$$\frac{1}{f(D)}(xV) = x\frac{1}{f(D)}V + \frac{d}{dD}\left(\frac{1}{f(D)}\right)V$$

where V is a function of x. We start with the fact that for a given function $g(x)$ we have

$$D(xg(x)) = xD(g(x)) + g(x)$$

Which can be rewritten as

$$D(xg(x)) = xD(g(x)) + \left(\frac{d}{dD}D\right)(g(x))$$

Operating D once more and after simplifications we obtain

$$D^2(xg(x)) = xD^2(g(x)) + \left(\frac{d}{dD}D^2\right)(g(x))$$

In general, by the method of induction for any natural number n we can show that

$$D^n(xg(x)) = xD^n(g(x)) + \left(\frac{d}{dD}D^n\right)g(x)$$

Direct implication of the above result leads

$$f(D)(xg(x)) = xf(D)(g(x)) + \left(\frac{d}{dD}f(D)\right)g(x) \tag{32.1}$$

Let us assume that $f(D)g(x) = V(x)$ so that we have

$$g(x) = \frac{1}{f(D)}V(x)$$

Substituting $g(x)$ in Equation (32.1) we get

$$f(D)\left(x\frac{1}{f(D)}V(x)\right) = xf(D)\left(\frac{1}{f(D)}V(x)\right) + \left(\frac{d}{dD}f(D)\right)\left(\frac{1}{f(D)}V(x)\right)$$

or

$$f(D)\left(x\frac{1}{f(D)}V(x)\right) = xV(x) + \left(\frac{d}{dD}f(D)\right)\left(\frac{1}{f(D)}V(x)\right)$$

This implies

$$xV(x) = f(D)\left(x\frac{1}{f(D)}V(x)\right) - \left(\frac{d}{dD}f(D)\right)\left(\frac{1}{f(D)}V(x)\right)$$

Operating the above equation by $1/f(D)$ we get

$$\frac{1}{f(D)}\left(xV(x)\right) = x\frac{1}{f(D)}V(x) - \left(\frac{d}{dD}f(D)\right)\left(\frac{1}{f(D)^2}V(x)\right)$$

Equivalently, we have the final result

$$\frac{1}{f(D)}\left(xV(x)\right) = x\frac{1}{f(D)}V(x) + \frac{d}{dD}\left(\frac{1}{f(D)}\right)V(x)$$

32.3.1 Example

Solve $(D^2+9)y = x\sin x$

Solution: The roots of the characteristic equations are $\pm 3i$. Hence, the complimentary function is given by

$$\text{C.F.} = (c_1\cos 3x + c_2\sin 3x)$$

The particular integral is

$$\text{P.I.} = \frac{1}{D^2+9}x\sin x$$

Using Rule V, we get

$$\text{P.I.} = x\frac{1}{D^2+9}\sin x + \frac{d}{dD}\left(\frac{1}{D^2+9}\right)\sin x$$

This can be now evaluated as

$$\text{P.I.} = x\frac{1}{8}\sin x - \frac{2D}{(D^2+9)^2}\sin x = \frac{1}{8}x\sin x - \frac{2D}{64}\sin x = \frac{1}{8}x\sin x - \frac{1}{32}\cos x$$

The required general solution is

$$y = (c_1\cos 3x + c_2\sin 3x) + \frac{1}{8}x\sin x - \frac{1}{32}\cos x$$

32.4 Rule VI: $F(x)$ is of the form $x^m \sin \alpha x$ or $x^m \cos \alpha x$

In this case Rule IV or Rule V can be applied. For the application of rule IV we should note that

1. $\frac{1}{f(D)} x^m \sin \alpha x = \text{Im}\left(\frac{1}{f(D)} x^m e^{i\alpha x}\right)$
2. $\frac{1}{f(D)} x^m \cos \alpha x = \text{Re}\left(\frac{1}{f(D)} x^m e^{i\alpha x}\right)$.

32.4.1 Example

Find a particular integral of $(D^2+1)y = x^2 \sin 2x$

Solution The particular integral is

$$\text{P.I.} = \frac{1}{D^2+1} x^2 \sin 2x = \text{Im} \frac{1}{D^2+1} x^2 e^{2ix}$$

Applying Rule IV, we get the particular integral as

$$\text{P.I.} = \text{Im}\left(e^{2ix} \frac{1}{(D+2i)^2+1} x^2\right) = \text{Im}\left(e^{2ix} \frac{1}{D^2+4Di-3} x^2\right)$$

$$\text{P.I.} = \text{Im}\left(\frac{e^{2ix}}{-3}\left[1-\left(\frac{4iD}{3}+\frac{D^2}{3}\right)\right]^{-1} x^2\right)$$

Using the Binomial expansion, we get

$$= \text{Im}\left(\frac{e^{2ix}}{-3}\left[1+\left(\frac{4iD}{3}+\frac{D^2}{3}\right)+\left(\frac{4iD}{3}+\frac{D^2}{3}\right)^2 \ldots\right] x^2\right)$$

$$= \text{Im}\left(\frac{e^{2ix}}{-3}\left[1+\frac{4iD}{3}+\frac{D^2}{3}-\frac{16D^2}{9}+\ldots\right] x^2\right)$$

$$= \text{Im}\left(-\frac{1}{3}(\cos 2x + i \sin 2x)\left[x^2 - \frac{8ix}{3} - \frac{26}{9}\right]\right)$$

Collecting the imaginary part we have

$$\text{P.I.} = -\frac{1}{3}\left[\left(x^2 - \frac{26}{9}\right)\sin 2x + \frac{8}{3} x \cos 2x\right].$$

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Lesson 33

Method of Undetermined Coefficients

In the last lesson we have discussed operator method of finding particular integral. In this lesson we lean method of undetermined coefficients for finding particular integral of non-homogeneous differential equations. This method is relatively easier to apply once a possible form of a particular integral is known. This method is mainly applicable to linear differential equations with constant coefficients.

33.1 Method of Undetermined Coefficients

The method of undetermined coefficients requires that we make an initial assumption about the form of a particular solution of the differential equation, but with the coefficients left unspecified. We then substitute the assumed expression into the given differential equation and attempt to determine the coefficients so as to satisfy that differential equation. If we are successful, then we have found a particular solution of the differential equation. If we cannot determine the coefficients, then this means that there is no solution of the form that we assumed. In this case we may modify the initial assumption and try again.

The main advantage of the method of undetermined coefficients is that it is straightforward to execute once the assumption is made as to the form of the particular solution. Its major limitation is that it is useful primarily for equations for which we can easily write down the correct form of the particular solution in advance. This method is usually used only for problems in which the homogeneous equation has constant coefficients and the nonhomogeneous term is restricted to a relatively small class of functions. In particular, we consider only nonhomogeneous terms that consist of polynomials, exponential functions, sines, and cosines. Despite this limitation, the method of undetermined coefficients is useful for solving many problems that have important applications.

We shall demonstrate the method by taking a couple of different examples.

33.2 Example Problems

33.2.1 Problem 1

Solve the following initial value problem

$$y'' + 5y' + 6y = 2x + 1 \tag{33.1}$$

with the initial conditions $y(0) = 0$ *and* $y'(0) = \frac{1}{3}$.

Solution: First we solve the corresponding homogeneous equation. The characteristic equation is

$$m^2 + 5m + 6 \quad \Rightarrow \quad m = -2, -3.$$

Hence the complementary function is

$$\text{C.F.} = C_1 e^{-2x} + C_2 e^{-3x}.$$

To find particular integral, the trick is to somehow to guess one particular solution to Equation (33.1). Note that $2x + 1$ is a polynomial, and the left hand side of the equation will be a polynomial if we let y be a polynomial of the same degree. Let us try

$$y_p = Ax + B$$

We plug in to the differential equation to obtain

$$\begin{aligned} y_p'' + 5y_p' + 6y_p =& (Ax + B)'' + 5(Ax + B)' + 6(Ax + B) \\ =& 0 + 5A + 6Ax + 6B = 6Ax + (5A + 6B). \end{aligned}$$

So $6Ax + (5A + 6B) = 2x + 1$. Therefore,

$$A = \frac{1}{3} \text{ and } B = \frac{-1}{9}$$

That means

$$y_p = \frac{1}{3}x - \frac{1}{9} = \frac{3x - 1}{9}$$

Hence the general solution to (33.1) is

$$y = C_1 e^{-2x} + C_2 e^{-3x} + \frac{3x - 1}{9}.$$

The general solution must satisfy the given initial conditions. First find

$$y' = -2C_1e^{-2x} - 3C_2e^{-3x} + \frac{1}{3}$$

Then

$$0 = y(0) = C_1 + C_2 - \frac{1}{9}, \qquad \frac{1}{3} = y'(0) = -2C_1 - 3C_2 + \frac{1}{3}.$$

We solve to get $C_1 = 1/3$ and $C_2 = -2/9$. The particular solution we want is

$$y(x) = \frac{1}{3}e^{-2x} - \frac{2}{9}e^{-3x} + \frac{3x-1}{9} = \frac{3e^{-2x} - 2e^{-3x} + 3x - 1}{9}.$$

33.2.2 Problem 2

Find a particular solution of the differential equation

$$y'' + 2y' + 2y = \cos(2x).$$

Solution: We start by guessing the solution that includes some multiple of $\cos(2x)$. We may have to also add a multiple of $\sin(2x)$ to our guess since derivatives of cosine are sines. We try

$$y_p = A\cos(2x) + B\sin(2x).$$

We plug y_p into the equation and we get

$$-4A\cos(2x) - 4B\sin(2x) - 4A\sin(2x) + 4B\cos(2x) + 2A\cos(2x) + 2B\sin(2x) = \cos(2x).$$

The left hand side must equal to right hand side. We group terms and get $-4A+4B+2A = 1$ and $-4B - 4A + 2B = 0$. So $-2A + 4B = 1$ and $2A + B = 0$ and hence $A = \frac{-1}{10}$ and $B = \frac{1}{5}$. Hence a particular solution is

$$y_p = A\cos(2x) + B\sin(2x) = \frac{-\cos(2x) + 2\sin(2x)}{10}.$$

Remark 1: *If the right hand side contains exponentials we try exponentials. For example, for*

$$Ly = e^{3x},$$

we will try $y = Ae^{3x}$ *as our guess and try to solve for* A.

Remark 2: *If the right hand side is a multiple of sines, cosines, exponentials, and polynomials, we can use the product rule for differentiation to come up with a guess. We need to guess a form for y_p such that Ly_p is of the same form, and has all the terms needed to for the right hand side. For example,*

$$Ly = (1 + 3x^2)\, e^{-x} \cos(\pi x).$$

For this equation, we will guess

$$y_p = (A + Bx + Cx^2)\, e^{-x} \cos(\pi x) + (D + Ex + Fx^2)\, e^{-x} \sin(\pi x).$$

We will plug in and then hopefully get equations that we can solve for A, B, C, D, E, and F.

Remark 3: *If the right hand side has several terms, such as*

$$Ly = e^{2x} + \cos x.$$

In this case we find u that solves $Lu = e^{2x}$ and v that solves $Lv = \cos x$ (that is, do each term separately). Then note that if $y = u + v$, then $Ly = e^{2x} + \cos x$. This is because L is linear; we have $Ly = L(u + v) = Lu + Lv = e^{2x} + \cos x$.

33.2.3 Problem 3

Find a particular solution of

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^t \cos 2t.$$

Solution: By splitting up the right side of the given differential equation, we obtain the three differential equations

$$y'' - 3y' - 4y = 3e^{2t}, \qquad y'' - 3y' - 4y = 2\sin t, \qquad y'' - 3y' - 4y = 8e^t \cos 2t$$

Solutions of these three equations can be found with appropriate guess of the particular integral discussed above. Finally, a particular solution is their sum, namely,

$$Y(t) = \frac{1}{2}e^{2t} + \frac{3}{17}\cos t \frac{5}{17}\sin t + \frac{10}{13}e^t cos2t + \frac{2}{13}e^t \sin 2t.$$

The procedure illustrated in these examples enables us to solve a large class of problems in a reasonably efficient manner. However, there is one difficulty that sometimes occurs. It could be that our guess actually solves the associated homogeneous equation. The next example illustrates how it arises.

33.2.4 Problem 4

Solve the following differential equation

$$y'' - 9y = e^{3x}$$

Solution: In order to find a particular integral an intelligent guess would be $y = Ae^{3x}$, but if we plug this into the left hand side of the equation we get

$$y'' - 9y = 9Ae^{3x} - 9Ae^{3x} = 0 \neq e^{3x}.$$

There is no way we can choose A to make the left hand side be e^{3x} because our guess satisfies homogeneous equation. Note that the general solution of the homogeneous equation is

$$\text{C.F.} = C_1e^{-3x} + C_2e^{3x}$$

Thus our assumed particular solution is actually a solution of the corresponding homogeneous equation; consequently, it cannot possibly be a solution of the nonhomogeneous equation. To find a particular solution we must therefore consider functions of a somewhat different form. We modify our guess to $y = Axe^{3x}$ and notice there is no difficulty anymore. Note that $y' = Ae^{3x} + 3Axe^{3x}$ and $y'' = 6Ae^{3x} + 9Axe^{3x}$. So

$$y'' - 9y = 6Ae^{3x} + 9Axe^{3x} - 9Axe^{3x} = 6Ae^{3x}.$$

Thus $6Ae^{3x}$ is supposed to equal e^{3x}. Hence, $6A = 1$ and so $A = \frac{1}{6}$. We can now write the general solution as

$$y = y_c + y_p = C_1e^{-3x} + C_2e^{3x} + \frac{1}{6}xe^{3x}.$$

33.2.5 Problem 5

Find a particular solution of

$$y'' + 4y = 3\cos 2t$$

Solution: First we write its complimentary function

$$\text{C.F.} = c_1 \cos 2t + c_2 \sin 2t$$

As in earlier example, we guess

$$y_p = At \cos 2t + Bt \sin 2t$$

Then, upon calculating y_p' and Y_p'', substituting them into the given differential equation, we find that

$$4A \sin 2t + 4B \cos 2t = 3cos2t$$

Therefore $A = 0$ and $B = 3/4$, so a particular solution of the given differential equation is

$$y_p(t) = \frac{3}{4} t \sin 2t$$

Remark 4: *It is also possible that multiplying by x does not get rid of the problem we had faced in last two examples. For example,*

$$y'' - 6y' + 9y = e^{3x}.$$

The complementary solution is $y_c = C_1 e^{3x} + C_2 x e^{3x}$. Guessing $y = Ae^{3x}$ or $y = Axe^{3x}$ would not get us anywhere. In this case we will guess $y_p = Ax^2 e^{3x}$.

Suggested Readings

Boyce, W.E. and DiPrima, R.C. (2001). Elementary Differential Equations and Boundary Value Problems. Seventh Edition, John Willey & Sons, Inc., New York.

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Lesson 34

Method of Variation of Parameters

In the last lesson we have discussed method of undetermined coefficients for finding particular integral. In this lesson we lean another rather general method, called method of variation of parameters, of finding particular integral of non-homogeneous differential equation. In contrast to the method of undetermined coefficients, this method is also applicable for solving linear equations with variable coefficients. For the sake of simplicity we restrict ourselves for second order linear differential equations. However the method is also applicable for higher order linear differential equations.

34.1 Method of Variation of Parameters

Consider a second order differential equation of the form

$$y'' + Py' + Qy = R \tag{34.1}$$

where P, Q, R are functions of x or constants. If u and v are two linearly independent solutions of the corresponding homogeneous differential equation

$$y'' + Py' + Qy = 0 \tag{34.2}$$

Then, the complimentary function is

$$y = au + bv \tag{34.3}$$

where a, b are two arbitrary constants and u, v are functions of x. Since u and v are solutions of (34.2), we have

$$u'' + Pu' + Qu = 0, \quad v'' + Pv' + Qv = 0. \tag{34.4}$$

The method of variation of parameters relies on finding a particular integral of nonhomogeneous equation by replacing constants a and b with functions of x. The aim is to find functions $A(x)$ and $B(x)$ such that

$$y_p = Au + Bv \tag{34.5}$$

is a particular integral of (34.1). To determine $A(x)$ and $B(x)$ we need to have two equations. These are obtained as follows. First we compute

$$y_p' = Au' + Bv' + A'u + B'v \tag{34.6}$$

In order to avoid second order derivatives of A and B and to simplify the above expression we take

$$A'u + B'v = 0, \tag{34.7}$$

Now, the Equation (34.6) reduces to

$$y_p' = Au' + Bv' \tag{34.8}$$

Differentiating (34.8), we obtain

$$y_p'' = A'u' + Au'' + B'v' + Bv'' \tag{34.9}$$

Using the values of y, y' and y'' given by (34.5), (34.8) and (34.9) into the Equation (34.1), we get

$$A'u' + Au'' + B'v' + Bv'' + P(Au' + Bv') + Q(Au + Bv) = R$$

Further simplifications lead to

$$A(u'' + Pu' + Qu) + B(v'' + Pv' + Qv) + A'u' + B'v' = R$$

Using Equation (34.4) we get

$$A'u' + B'v' = R \tag{34.10}$$

Solving (34.7) and (34.10) for A' and B', we get

$$A' = \frac{-vR}{W}, \quad \text{and} \quad B' = \frac{uR}{W} \tag{34.11}$$

where W is Wronskian of u and v, and given by $W = uv' - u'v \neq 0$. Note that the Wronskian is nonzero because u and v are two linearly independent solutions. Integrating (34.11), we get

$$A = f(x), \quad B = g(x), \tag{34.12}$$

where

$$f(x) = -\int \frac{vR}{W}dx, \quad g(x) = \int \frac{uR}{W}dx$$

Using (34.12) into (34.5), we have

$$y_p = uf(x) + vg(x).$$

Hence, the general solution of the given differential equation is

$$y_p = au + bv + uf(x) + vg(x).$$

34.2 Example Problems

34.2.1 Problem 1

Solve the differential equation $y'' + n^2y = \sec nx$.

Solution: Comparing the given equation with the standard equation $y'' + Py' + Qy = R$, we get $P = 0$, $Q = n^2$ and $R = \sec nx$. The characteristic equation of the corresponding homogeneous equation is

$$(m^2 + n^2)y = 0, \text{ so that } m = \pm in$$

The complimentary function is

$$\text{C.F.} = (c_1 \cos nx + c_2 \sin nx)$$

In order to find a particular integral, we have $u = \cos nx$ and $v = \sin nx$ and $R = \sec nx$. The Wronskian is given as

$$W = \begin{vmatrix} \cos nx & \sin nx \\ -n\sin nx & n\cos nx \end{vmatrix} = n \neq 0.$$

Then, the particular integral of the given equation is

$$\text{P.I.} = uf(x) + vg(x)$$

where

$$f(x) = -\int \frac{vR}{W}dx = -\int \frac{\sin nx \sec nx}{n}dx = \frac{1}{n^2}\ln(\cos nx)$$

and

$$g(x) = \int \frac{uR}{W} dx = -\int \frac{\cos nx \sec nx}{n} dx = \frac{x}{n}$$

Hence, the required solution is

$$y = (c_1 \cos nx + c_2 \sin nx) + \cos nx \frac{1}{n^2} \ln(\cos nx) + \frac{x}{n} \sin nx$$

34.2.2 Problem 2

Find the general solution of the differential equation $y'' + n^2 y = \tan nx$.

Solution: We compare the given equation with $y'' + Py' + Qy = R$ to have $P = 0$, $Q = n^2$ and $R = \tan nx$. Similar to the previous example, we have the complimentary function as

$$\text{C.F.} = (c_1 \cos nx + c_2 \sin nx)$$

To find particular integral we have $u = \cos nx$, $v = \sin nx$, $R = \sec nx$. The Wronskian is given by

$$W = \begin{vmatrix} \cos nx & \sin nx \\ -n \sin nx & n \cos nx \end{vmatrix} = n \neq 0.$$

The particular integral is

$$\text{P.I.} = uf(x) + vg(x)$$

where

$$f(x) = -\int \frac{vR}{W} dx = -\int \frac{\sin nx \tan nx}{n} dx = \frac{1}{n^2}[\sin nx - \ln(\sec nx + \tan nx)]$$

and

$$g(x) = \int \frac{uR}{W} dx = -\int \frac{\cos nx \tan nx}{n} dx = -\frac{1}{n^2} \cos nx$$

The desired general solution is

$$y = (c_1 \cos nx + c_2 \sin nx) + \frac{\cos nx}{n^2}[\sin nx - \ln(\sec nx + \tan nx)] - \frac{1}{n^2} \sin nx \cos nx$$

34.2.3 Problem 3

Solve the differential equation $\frac{d^2y}{dx^2} + n^2 y = \cot nx$.

Solution: Similar to the previous example, the complimentary function is given by

$$\text{C.F.} = (c_1 \cos nx + c_2 \sin nx)$$

In this case, we have $u = \cos nx$, $v = \sin nx$ and $R = \cot nx$. The Wronskian is given by

$$W = \begin{vmatrix} \cos nx & \sin nx \\ -n \sin nx & n \cos nx \end{vmatrix} = n \neq 0$$

Then, the particular integral is

$$\text{P.I.} = uf(x) + vg(x)$$

where

$$f(x) = -\int \frac{vR}{W} dx = -\int \frac{\sin nx \cot nx}{n} dx = -\frac{1}{n^2} \sin nx$$

and

$$g(x) = \int \frac{uR}{W} dx = -\int \frac{\cos nx \cot nx}{n} dx = \frac{1}{n^2} \left[\cos nx + \ln \left(\tan \frac{nx}{2} \right) \right]$$

The required solution is

$$y = (c_1 \cos nx + c_2 \sin nx) - \frac{1}{n^2} \cos nx \sin nx + \frac{1}{n^2} \left[\cos nx + \ln \left(\tan \frac{nx}{2} \right) \right] \sin nx$$

34.2.4 Problem 4

Using the method of variation of parameters, find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + y = \sec^2 x$$

Solution: The complimentary function is given by

$$\text{C.F.} = (c_1 \cos x + c_2 \sin x)$$

Also we have $u = \cos x$ and $v = \sin x$ and $R = \sec^2 x$ and

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1 \neq 0.$$

Then, the particular integral is

$$\text{P.I.} = uf(x) + vg(x)$$

where

$$f(x) = -\int \frac{vR}{W}dx = -\int \sin x \sec^2 x dx = -\int \sec x \tan x dx = -\sec x$$

and

$$g(x) = \int \frac{uR}{W}dx = \int \cos x \sec^2 x dx = \int \sec x dx = \ln[\sec x + \tan x]$$

The required solution is

$$y = (c_1 \cos x + c_2 \sin x) - \cos x \ \sec x + \sin x \ln[\sec x + \tan x].$$

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Lesson 35

Equations Reducible to Linear Differential Equations with Constant Coefficients

In this lesson we shall study two special forms of linear equations with variable coefficients which can be reduced to linear differential equations with constant coefficients by a suitable substitution. Those special forms which we study here are called Cauchy-Euler homogeneous linear differential equations and Legendre's homogeneous linear differential equations.

35.1 Cauchy-Euler Homogeneous Linear Differential Equation

A linear differential equation of the form

$$a_0x^n\frac{d^ny}{dx^n} + a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_2x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_ny = F(x), \tag{35.1}$$

where $a_1, a_2, \ldots, a_n$ are constants and F is either a constant or a function of x only, is called Cauchy-Euler homogeneous linear differential equation. Note that the index of x and order of derivative is same in each term of such equations.

Using the symbols $D(= d/dx), D^2(= d^2/dx^2), \ldots, D^n(= d^n/dx^n)$, the Equation (35.1) becomes

$$(a_0x^nD^n + a_1x^{n-1}D^{n-1} + a_2x^{n-2}D^{n-2} + \ldots + a_n)y = F(x) \tag{35.2}$$

The above equation can be reduced to linear differential equation with constant coefficients by substituting

$$x = e^z, \quad \text{or} \quad \ln x = z, \quad \text{so that} \quad \frac{dz}{dx} = \frac{1}{x} \tag{35.3}$$

Using chain rule for differentiation we obtain

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$$

Defining $\frac{d}{dz} =: D_1$, we have

$$x\frac{dy}{dx} = \frac{dy}{dz} \quad \Leftrightarrow \quad xDy = D_1 y$$

Similarly, for the second order derivative

$$\begin{aligned}\frac{d^2y}{dx^2} =& \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right) \\ =& -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}\end{aligned}$$

Thus, we have

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} \quad \Rightarrow \quad x^2D^2y = D_1(D_1-1)y.$$

Similarly, $x^3D^3y = D_1(D_1-1)(D_1-2)y$ and so on. In general, we have the relationship

$$x^nD^n = D_1(D_1-1)(D_1-2)\ldots(D_1-n+1)y$$

Substituting the above values of $x, xD, x^2D^2, \ldots, x^nD^n$ in the Equation (35.1), we get

$$[a_0D_1(D_1-1)\ldots(D_1-n+1) + \ldots + a_{n-2}D_1(D_1-1) + a_{n-1}D_1 + a_n]\,y = F(e^z) \quad (35.4)$$

The Equation (35.4) is a linear differential equation with constant coefficients which can solved with the methods discussed in previous lessons. Finally, by replacing z by $\ln x$ we obtain the desired solution of the given differential equation.

35.2 Example Problems

35.2.1 Problem 1

Solve the differential equation $(x^2D^2 + xD - 4)y = 0$.

Solution: Substituting $x = e^z \;\Rightarrow\; \ln x = z \;\Rightarrow\; xD = D_1$, $x^2D^2 = D_1(D_1-1)$, the given equation reduces to

$$[D_1(D_1-1) + D_1 - 4]\,y = 0 \quad \Rightarrow \quad (D_1^2 - 4)y = 0$$

The roots of the corresponding characteristic equation are $m = 2, -2$. The required solution of the transformed equation is

$$y = c_1 e^{2z} + c_2 e^{-2z}$$

Putting $\log x = z$, we have the desired solution as

$$y = c_1 x^2 + c_2 x^{-2}.$$

Here c_1 and c_2 are arbitrary constants.

35.2.2 Problem 2

Find the general solution of the differential equation $(x^2 D^2 + y)y = 3x^2$.

Solution: Substituting $x = e^z$, the given equation reduces to

$$(D_1(D_1 - 1) + 1)y = 3e^{2z} \quad \Rightarrow \quad (D_1^2 - D_1 + 1)y = 3e^{2z}$$

The characteristic equation of this differential equation is

$$(m^2 - m + 1) = 0 \quad \Rightarrow \quad m = (1 \pm i\sqrt{3})/2$$

The complimentary function is

$$\text{C.F.} = e^{z/2}\left[c_1 \cos\left(z\sqrt{3}/2\right) + \left(c_1 \sin z\sqrt{3}/2\right)\right]$$

Substituting $z = \ln x$, we get

$$\text{C.F.} = \sqrt{x}\left[c_1 \cos\left(\ln x\sqrt{3}/2\right) + c_1 \sin\left(\ln x\sqrt{3}/2\right)\right]$$

The particular integral of the transformed equation is

$$\text{P.I.} = \frac{1}{D_1^2 - D_1 + 1} 3e^{2z} = \frac{1}{2^2 - 2 + 1} 3e^{2z} = e^{2z}$$

Hence, the desired solution of the given differential equation is

$$y = \sqrt{x}\left[c_1 \cos\left(\ln x\sqrt{3}/2\right) + c_1 \sin\left(\ln x\sqrt{3}/2\right)\right] + x^2$$

35.3 Legendre's Homogeneous Linear Differential Equations

A linear differential equation of the form is

$$[(a+bx)^n a_0 D^n + a_1(a+bx)^{n-1}D^{n-1} + a_2(a+bx)^{n-2}D^{n-2} + ... + a_n]y = F(x), \quad (35.5)$$

where $a, b, a_1, a_2, ..., a_n$ are constants, and F is either a constant or a function of x only, is called a Legendre's homogeneous linear differential equation. Note that the index of $(a+bx)$ and the order of derivative is same in each term of such equation. To solve the Equation (35.5), we introduce a new independent variable z such that

$$a + bx = e^z, \quad \text{or} \quad \ln(a+bx) = z, \quad \text{so that} \quad b/(a+bx) = dz/dx. \quad (35.6)$$

Now, for the first order derivative we have

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{b}{(a+bx)}\frac{dy}{dz}$$

This implies

$$(a+bx)\frac{dy}{dx} = b\frac{dy}{dz} \quad \Leftrightarrow \quad (a+bx)Dy = bD_1 y$$

Similarly for the second order derivative we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{b}{(a+bx)}\frac{dy}{dz}\right)$$

This can be further simplified to get

$$\begin{aligned}\frac{d^2y}{dx^2} &= -\frac{b^2}{(a+bx)^2}\frac{dy}{dz} + \frac{b}{(a+bx)}\frac{d}{dx}\left(\frac{dy}{dz}\right) \\ &= -\frac{b^2}{(a+bx)^2}\frac{dy}{dz} + \frac{b}{(a+bx)}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}\end{aligned}$$

Substituting dz/dx from Equation (35.6), we obtain

$$\frac{d^2y}{dx^2} = -\frac{b^2}{(a+bx)^2}\frac{dy}{dz} + \frac{b^2}{(a+bx)^2}\frac{d^2y}{dz^2}$$

This gives us

$$(a+bx)^2\frac{d^2y}{dx^2} = b^2\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) \quad \Leftrightarrow \quad (a+bx)^2 D^2 y = b^2 D_1(D_1 - 1)y$$

In general, we have

$$(a+bx)^n D^n = b^n D_1(D_1-1)(D_1-2)\dots(D_1-n+1)y$$

Substituting the above values of $(a+bx), (a+bx)D, (a+bx)^2D^2, \dots, (a+bx)^nD^n$ in the Equation (35.5), we get the following linear differential equation with constant coefficients

$$\left[a_0 b^n D_1(D_1-1)...(D_1-n+1) + ... + a_{n-2}b^2 D_1(D_1-1) + a_{n-1}bD_1 + a_n\right] y = F\left(\frac{e^z-a}{b}\right)$$

The methods of solving this transformed equation are same as discussed in previous section.

35.3.1 Example

Solve the differential equation

$$(1+x)^4\frac{d^3y}{dx^3} + 2(1+x)^3\frac{d^2y}{dx^2} - (1+x)^2\frac{dy}{dx} + (1+x)y = \frac{1}{(1+x)}$$

Solution: Using $D = \frac{d}{dx}$ and dividing both sides by $(x+1)$, the given differential equation can be rewritten as

$$\left[(1+x)^3D^3 + 2(1+x)^2D^2 - (1+x)D + 1\right] y = (1+x)^{-2}.$$

This is the Legendre's homogeneous linear equation which can be solved by substituting

$$(1+x) = e^z \Leftrightarrow \ln(1+x) = z$$

This substitution readily implies

$$(1+x)D = D_1, \quad (1+x)^2D^2 = D_1(D_1-1), \quad (1+x)^3D^3 = D_1(D_1-1)(D_1-2)$$

The given differential equation reduces to

$$[D_1(D_1-1)(D_1-2) + 2D_1(D_1-1) - D_1 + 1]\, y = e^{-2z}$$

or

$$\left(D_1^3 - D_1^2 - D_1 + 1\right) y = e^{-2z}$$

The characteristic equation of the corresponding homogeneous equation is

$$\left(m^3 - m^2 - m + 1\right) y = 0$$

The roots of the characteristics equations are $m = 1, 1, -1$. Hence the complimentary function of the transformed differential equation is

$$\text{C.F.} = (c_1 + c_2 z)e^z + c_3 e^{-z}$$

The particular integral of the transformed differential equation can be found as

$$\begin{aligned}\text{P.I.} =& \frac{1}{(D_1^3 - D_1^2 - D_1 + 1)} e^{-2z} \\ =& \frac{1}{-2^3 - 2^2 + 2 + 1} e^{-2z} \\ =& -\frac{1}{9} e^{-2z}\end{aligned}$$

Hence the general solution of the transformed differential equation is

$$y = (c_1 + c_2 z)e^z + c_3 e^{-z} - \frac{1}{9} e^{-2z}$$

Replacing z by $\ln(1 + x)$ we obtain the desired solution of the given differential equation

$$y = [c_1 + c_2 \ln(1 + x)]\,(1 + x) + \frac{c_3}{(1 + x)} - \frac{1}{9}\frac{1}{(1 + x)}.$$

Suggested Readings

Boyce, W.E. and DiPrima, R.C. (2001). Elementary Differential Equations and Boundary Value Problems. Seventh Edition, John Willey & Sons, Inc., New York.

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Lesson 36

Methods for Solving Simultaneous Ordinary Differential Equations

In this lesson we shall consider systems of simultaneous linear differential equations which contain a single independent variable and two or more dependent variables. We will consider two different techniques, mainly the method of elimination and the method of differentiation, for solving system of differential equations.

36.1 Simultaneous Ordinary Linear Differential Equations

Let x and y be the dependent and t be the independent variable. Thus, in such equations there occur differential coefficients of x, y with respect to t. Let $D = d/dt$, then such equations can be put into the form

$$f_1(D)x + f_2(D)y = T_1 \tag{36.1}$$

$$g_1(D)x + g_2(D)y = T_2 \tag{36.2}$$

where T_1 and T_2 are functions of the independent variable t and $f_1(D), f_2(D), g_1(D)$, and $g_2(D)$ are all rational integral functions of D with constant coefficients. In general, the number of equations will be equal to the number of dependent variables, i.e., if there are n dependent variables there will be n equations.

36.2 Method of Elimination

In order to eliminate y between equations (36.1) and (36.2), operating on both sides of (36.1) by $g_2(D)$ and on both sides of (36.2) by $f_2(D)$ and subtracting, we get

$$(f_1(D)g_2(D) - g_1(D)f_2(D))\, x = g_2(D)T_1 - f_2(D)T_2 \tag{36.3}$$

This is a linear differential equation with constant coefficients in x and t and can be solved to give the value of x in terms of t. Substituting this value of x in either (36.1) or (36.2), we get the value of y in terms of t.

Remark 1: *The above Equations* (36.1) *and* (36.2) *can be also solved by first eliminating* x *between them and solving the resulting equation to get* y *in terms of* t*. Substituting this value of* y *in either* (36.1) *or* (36.2)*, we get the value of* x *in terms of* t*.*

Remark 2: *In the general solutions of* (36.1) *and* (36.2) *the number of arbitrary constants will be equal to the sum of the orders of the equations* (36.1) *and* (36.2)*.*

36.3 Example Problems

36.3.1 Problem 1

Solve the simultaneous equations

$$\frac{dx}{dt} - 7x + y = 0 \tag{36.4}$$

$$\frac{dy}{dt} - 2x - 5y = 0 \tag{36.5}$$

Solution: Writing D for d/dt, the given equations can be rewritten in the following symbolic form as

$$(D-7)x + y = 0 \tag{36.6}$$

$$-2x + (D-5)y = 0 \tag{36.7}$$

Now, we eliminate x by multiplying Equation (36.6) by 2 and operating (36.7) by $(D-7)$ as follows

$$2(D-7)x + 2y = 0 \tag{36.8}$$

$$-2(D-7)x + (D-7)(D-5)y = 0 \tag{36.9}$$

Adding (36.8) and (36.9), we get

$$[(D-7)(D-5) + 2]y = 0$$

or

$$(D^2 - 12D + 37)y = 0$$

This is a linear equation with constants coefficients. Its auxiliary equation is

$$(m^2 - 12m + 37) = 0$$

The roots of the auxiliary equation are $m = 6 \pm i$. Therefore, we get the general solution for the variable y as

$$y = e^{6t}(c_1 \cos t + c_2 \sin t), \tag{36.10}$$

where c_1 and c_2 being arbitrary constants. We now find x by using Equation (36.7). Now from (36.10), differentiating w.r.t. t, we get

$$Dy = 6e^{6t}[(c_1 \cos t + c_2 \sin t) + e^{6t}(-c_1 \sin t + c_2 \cos t)],$$

or on simplifications we obtain

$$Dy = 6e^{6t}[(6c_1 + c_2) \cos t + (-c_1 + 6c_2) \sin t] \tag{36.11}$$

Now, substituting y and Dy in the Equation (36.7), we get

$$x = (1/2) \times e^{6t}[(c_1 + c_2) \cos t + (-c_1 + c_2) \sin t] \tag{36.12}$$

Thus, equations (36.10) and (36.12) give the desired general solution.

36.3.2 Problem 2

Solve the linear system of differential equations

$$D^2y - y + 5Dv = x \tag{36.13}$$

$$2Dy - D^2v + 4v = 2 \tag{36.14}$$

Solution: Multiplying (36.13) by $2D$ and (36.14) by $(D^2 - 1)$ and then subtracting (36.14) from the Equation (36.13) we obtain

$$[10D^2 + (D^2 - 1)(D^2 - 4)]v = 2Dx - (D^2 - 1)2$$

or

$$(D^4 + 5D^2 + 4)v = 4 \tag{36.15}$$

This is a linear differential equations with constant coefficients whose solution can easily be found. The characteristic equation of the corresponding homogeneous equation is

$$m^4 + 5m^2 + 4 = 0 \quad \Rightarrow \quad (m^2+1)(m^2+4) = 0 \quad \Rightarrow \quad m = \pm i, \pm 2i$$

The complimentary function is

$$\text{C.F.} = c_1 \cos x + + c_2 \sin x + c_3 \cos 2x + c_4 \sin 2x$$

The particular integral is

$$\text{P.I.} = \frac{1}{D^4 + 5D^2 + 4} 4e^{0x} = 1$$

We write the general solution for v as

$$v = 1 + c_1 \cos x + c_2 \sin x + c_3 \cos 2x + c_4 \sin 2x \tag{36.16}$$

Now we find an equation giving y in terms of v. This can be done by eliminating from the equations (36.13) and (36.14) those terms which involve derivatives of y. So multiplying Equation (36.13) by 2 and Equation (36.14) by D we get

$$(2D^2 - 2)y + 10Dv = 2x \tag{36.17}$$

$$2D^2 y - (D^3 - 4D)v = 0 \tag{36.18}$$

Subtracting (36.17) from (36.18) we get

$$2y - D^3 v - 6Dv = -2x \tag{36.19}$$

or

$$y = -x + \frac{1}{2} D^3 v + 3Dv \tag{36.20}$$

Substitute v from (36.16) into the Equation (36.21) to obtain the expression for y as

$$y = -x - \frac{5}{2} c_1 \cos x + \frac{5}{2} c_2 \cos x + 2c_4 \cos 4x - 2c_3 \sin 2x, \tag{36.21}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants.

36.4 Method of Differentiation

Sometimes, x and y can be eliminated if we differentiate (36.1) or (36.2). For example, assume that the given equations (36.1) and (36.2) relates four quantities $x, y, dx/dt$ and dy/dt. Differentiating (36.1) and (36.2) with respect to t, we obtain four equations containing $x, dx/dt, d^2x/dt^2, y, dy/dt$ and d^2y/dt^2. Eliminating three quantities $y, dy/dt$ and d^2y/dt^2 from these four equations, y is eliminated and we get an equation of the second order with x as the dependent and t as the independent variable. Solving this equation we get value of x in terms of t. Substituting this value of x in either (36.1) or (36.2), we get value of y in terms of t. The technique will be illustrated by the following example.

36.4.1 Example

Determine the general solutions for x and y for

$$\frac{dx}{dt} - y = t$$
$$\frac{dy}{dt} + x = 1$$

Solution: Writing D for d/dt, the given equations become

$$Dx - y = t \tag{36.22}$$
$$x + Dy = 1 \tag{36.23}$$

Differentiating the equation Equation (36.22) w.r.t. t we get

$$D^2x - Dy = 1 \tag{36.24}$$

Now we can eliminate y by adding equations (36.24) and (36.23) to get

$$(D^2 + 1)x = 2 \tag{36.25}$$

The auxiliary equation of the above differential equation is $m^2 + 1 = 0$ and therefore the general solution of the homogeneous equation is

$$\text{C.F.} = c_1 \cos t + c_2 \sin t$$

where c_1 and c_2 are arbitrary constants. The particular integral is

$$\text{P.I} = \frac{1}{D^2+1}2 = (1+D^2)^{-1}2 = (1-D^2+...)2 = 2$$

Hence, the general solution of (36.25) is

$$x = c_1 \cos t + c_2 \sin t + 2 \tag{36.26}$$

From Equation (36.22), we get

$$y = c_2 \cos t - c_1 \sin t - t \tag{36.27}$$

Thus, the required solution is given by (36.26) and (36.27).

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Lesson 37

Series Solutions about an Ordinary Point

37.1 Introduction

If we can't find a solution to a differential equations in a form of nice functions, we can still look for a series representation of the solution. Series solutions are very useful because if we know that the series converges, we can approximate the solution as closely as we want. In this lesson we describe series solutions of solving second order linear homogeneous differential equations with variables coefficients. Series solution can be used in conjunction with variation of parameters to solve linear nonhomogeneous equations. For simplicity, we shall be dealing mainly with polynomial coefficients. Here we consider the second order homogeneous equation of the form

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{37.1}$$

where P, Q and R are polynomials or analytic functions in general. Many problems in mathematical physics leads to equations of the form (37.1) having polynomial coefficients; for example, the Bessel equation

$$x^2y'' + xy' + (x^2a^2)y = 0,$$

where a is a constant, and the Legendre equation

$$(1-x)^2y'' - 2xy' + c(c+1)y = 0$$

where c is a constant.

37.2 Useful Definitions

Here we provide some definitions which will be very useful for finding series solution of the differential equations.

37.2.1 Analytic Function

A function $f(x)$ defined on an interval containing the point $x = x_0$ is called analytic at x_0 if its Taylor series,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0) \tag{37.2}$$

exists and converges to $f(x)$ for all x in the interval of convergence of (37.2).

37.2.2 Ordinary Points

A point $x = x_0$ is called an ordinary point of the Equation (37.1) if P, Q, and R are polynomials that do not have any common factors, then a point x_0 is called an ordinary point if $P(x_0) \neq 0$. A point x_1 where $P(x_1) = 0$ is called a singular point. If any of P, Q, or R is not a polynomial, then we call x_0 an ordinary point if $Q(x)/P(x)$ and $R(x)/P(x)$ are analytic about x_0.

It is often useful to rewrite Equation (37.1) as

$$y'' + p(x)y' + q(x)y = 0 \tag{37.3}$$

where $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$. The Equation (37.3) is called equivalent normalized form of the Equation (37.1).

37.2.3 Singular Points

If the point $x = x_0$ is not an ordinary point of the differential Equation (37.1) or (37.3), then it is called a singular point of the differential equation of (37.3). There are two types of singular points: (i) regular singular points, and (ii) irregular singular points. A singular point $x = x_0$ of the differential Equation (37.3) is called a regular singular point of the differential Equation (37.3) if both

$$(x - x_0)p(x) \text{ and } (x - x_0)^2 q(x)$$

are analytic at $x = x_0$. A singular point, which is not regular is called an irregular singular point.

37.3 Example Problems

37.3.1 Problem 1

Show that $x = 0$ is an ordinary point of $(x^2 - 1)y'' + xy' - y = 0$, but $x = 1$ is a regular singular point.

Solution: Writing the given equation in normalized form

$$\frac{d^2y}{dx^2} + \frac{x}{(x-1)(x+1)}\frac{dy}{dx} - \frac{1}{(x-1)(x+1)}y = 0. \tag{37.4}$$

Comparing (37.4) with the standard equation $y'' + p(x)y' + q(x)y = 0$, we have

$$p(x) = x/(x-1)(x+1) \text{ and } q(x) = -1/(x-1)(x+1).$$

Since both $p(x)$ and $q(x)$ are analytic at $x = 0$, the point $x = 0$ is an ordinary point of the given Equation (37.4). Further note that both $p(x)$ and $q(x)$ are not analytic at $x = 1$, thus $x = 1$ is not an ordinary point and so $x = 1$ is a singular point. Also

$$(x-1)P(x) = x/(x+1) \text{ and } (x-1)^2Q(x) = -(x-1)/(x+1)$$

show that both $(x-1)P(x)$ and $(x-1)^2Q(x)$ are analytic at $x = 1$. Therefore $x = 1$ is a regular singular point.

37.3.2 Problem 2

Determine whether the point $x = 0$ is an ordinary point or regular point of the differential equation

$$xy'' + \sin(x)y + x^2y = 0$$

Solution: Comparing with the normalized equation we get

$$p(x) = \frac{\sin x}{x} \text{ and } q(x) = x$$

Since $p(x)$ and $q(x)$ both are analytic at $x = 0$, the point $x = 0$ is an ordinary point. This example shows that singular point does not always occur where $P(x) = 0$.

37.3.3 Problem 3

Discuss the singular points of the differential equation

$$x^2(x-2)^2y'' + (x-2)y' + 3x^2y = 0.$$

Solution: Clearly the function

$$p(x) = \frac{1}{(x^2(x-2))}$$

is not analytic at $x = 0$ and $x = 2$. Also the function

$$q(x) = \frac{3}{((x-2)^2)}$$

is not analytic at $x = 2$. Hence both $x = 0$ and $x = 2$ are singular point of the differential equations. At $x = 0$ we have

$$xp(x) = \frac{1}{(x(x-2))} \quad \text{and} \quad x^2q(x) = \frac{3x^2}{(x-2)^2}$$

Note that $x^2q(x)$ is non-singular at $x = 0$ but $xp(x)$ is not analytic at this point. Hence $x = 0$ is an irregular singular point. At $x = 2$ we have

$$(x-2)p(x) = \frac{1}{x^2} \quad \text{and} \quad (x-2)^2q(x) = 2$$

Both functions are analytic at $x = 2$ and hence $x = 2$ is a regular singular point.

37.4 Brief Overview of Power Series

A power series about a point x_0 is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-x_0)^n$$

where x is a variable and c_n are constants, called coefficients of the series. There are three possibilities about the convergence of a power series. The series may converge only at $x = 0$ or it may converge for all values of x. If this is not the case then a definite positive number R exists such that the given series converges for every $|x - x_0| < R$ and

diverges for every $|x-x_0| > R$. Such a number is known as the radius of convergence and $]x_0 - R, x_0 + R[$, the interval of convergence, of the given series.

Among several formulas for determining convergence of the power series, ratio test is most common and simple to use. Given a power series $\sum_{n=0}^{\infty} c_n(x-x_0)^n$ we compute

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|,$$

then the series is convergence for $|x-x_0| < R$ and divergent $|x-x_0| > R$.

37.4.1 Example

Determine the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x+1)^n}{n2^n}$$

Solution: Ratio test gives

$$\lim_{n\to\infty} \left| \frac{n2^n}{(n+1)2^{n+1}} \right| = \frac{1}{2}$$

Hence the radius of convergence of the power series is $R = 2$ and the interval of convergence is $-3 < x < 1$. The convergence at the end points $x = -3$ and $x = 1$ needs to be checked separately.

37.5 Power Series Solution near Ordinary Point

Let the given equation be

$$y'' + p(x)y' + q(x)y = 0 \tag{37.5}$$

If $x = x_0$ is an ordinary point of (37.5), then (37.5) has two non-trivial linearly independent power series solutions of the form

$$\sum_{n=0}^{\infty} C_n(x-x_0)^n \tag{37.6}$$

and these power series converge in some interval of convergence $|x-x_0| < R$, (where R is the radius of convergence of (37.6)) about x_0.

To find series solutions we suppose that we have a series representation,

$$y = \sum_{n=0}^{\infty} C_n (x - x_0)^n \tag{37.7}$$

and then to find out coefficients C_n we need to differentiate (37.7) and plug in the derivatives into the Equation (37.6). Once we have the appropriate coefficients, we call (37.7) the series solution to (37.5) near $x = x_0$. More precisely, differentiating twice, the Equation (37.7) yields

$$y' = \sum_{n=0}^{\infty} n C_n (x - x_0)^{n-1} \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} n(n-1) C_n (x - x_0)^{n-2} \tag{37.8}$$

Substituting the above values of y, y' and y'' in (37.5), we obtain

$$A_0 + A_1(x - x_0) + A_2(x - x_0)^2 + \ldots + A_n(x - x_0)^n + \ldots = 0, \tag{37.9}$$

where the coefficients $A_0, A_1, A_2 \ldots$ etc. are now some functions of the coefficients $C_0, C_1, C_2, \ldots$ etc. Since the Equation (37.9) is an identity, all the coefficients $A_0, A_1, A_2 \ldots$ of (37.9) must be zero, i.e.,

$$A_0 = 0, A_1 = 0, A_2 = 0, \ldots, A_n = 0 \tag{37.10}$$

Solving Equation (37.10), we obtain the coefficients of (37.7) in terms of C_0 and C_1. Substituting these coefficients in (37.7), we obtain the required series solution of (37.5) in power of $(x - x_0)$.

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Lesson 38

Series Solution about an Ordinary Point (Cont.)

In the last lesson we have discussed series solution of the homogeneous differential equations. In this lesson we demonstrate the method by using a couple of basic examples. For demonstration we take first example of a differential equation with constant coefficients and then some more involved examples will be discussed.

38.1 Example Problems

38.1.1 Problem 1

Determine a series solution to $y'' - y = 0$.

Solution: Suppose that the series solution is of the form

$$y(x) = \sum_{n=0}^{\infty} c_n x_n$$

Differentiating y, we have

$$y'(x) = \sum_{n=1}^{\infty} n c_n x_{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) c_n x_{n-2}$$

Substituting these into the differential equation, we have

$$\sum_{n=2}^{\infty} n(n-1) c_n x_{n-2} - \sum_{n=0}^{\infty} c_n x_n = 0$$

Re-indexing the first sum

$$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x_n - \sum_{n=0}^{\infty} c_n x_n = 0$$

This implies

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) c_{n+2} - c_n x_n \right] x_n = 0$$

Since the series is always equal to 0 then each coefficient must be zero. Thus we have

$$(n+2)(n+1)c_{n+2} - c_n = 0 \tag{38.1}$$

This can be rewritten in the form of recurrence relation as

$$c_{n+2} = \frac{c_n}{(n+2)(n+1)} \tag{38.2}$$

Putting $n = 0, 1, 2 \ldots$, we get

$$c_2 = \frac{c_0}{2!}, \quad c_3 = \frac{c_1}{3!}, \quad c_4 = \frac{c_0}{4!}, \quad c_5 = \frac{c_1}{5!}, \ldots$$

In general, we have

$$c_{2k} = \frac{c_0}{(2k)!}, \quad c_{2k+1} = \frac{c_1}{(2k+1)!} \ldots \text{ for } k = 1, 2, \ldots.$$

Putting these values into the series and collecting the c_0 and c_1 terms we get

$$y(x) = c_0 \left(1 + \frac{x^2}{2!} + \ldots + \frac{x^{2k}}{(2k)!} + \ldots\right) + c_1 \left(x + \frac{x^3}{3!} + \ldots + \frac{x^{2k+1}}{(2k+1)!} + \ldots\right)$$

This can be further rewritten in summation form as

$$y(x) = c_0 \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} + c_1 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

This is the desired series solution. It should be noted that this series solution can be rewritten into the form of well known solution $y(x) = \overline{c}_1 e^x + \overline{c}_2 e^{-x}$ of the given differential equation as

$$\overline{c}_1 e^x + \overline{c}_2 e^{-x} = \overline{c}_1 \left(1 + x + \frac{x^2}{2!} + \ldots\right) + \overline{c}_2 \left(1 - x + \frac{x^2}{2!} + \ldots\right)$$

This can be rewritten as

$$\overline{c}_1 e^x + \overline{c}_2 e^{-x} = (\overline{c}_1 + \overline{c}_2) \left(1 + \frac{x^2}{2!} + \ldots\right) + (\overline{c}_1 - \overline{c}_2) \left(x + \frac{x^3}{3!} + \ldots\right)$$

Denoting $(\overline{c}_1 + \overline{c}_2) =: c_0$ and $(\overline{c}_1 - \overline{c}_2) =: c_1$ we get

$$\overline{c}_1 e^x + \overline{c}_2 e^{-x} = c_0 \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} + c_1 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

This proves that both representations are equivalent.

38.1.2 Problem 2

Find the series solution, about $x = 0$*, of the equation* $(1-x)^2y'' - 2y = 0$ *in powers of* x*.*

Solution: Since $x = 0$ is an ordinary point and we can therefore get two linearly independent solution by substituting

$$y = \sum_{n=0}^{\infty} c_n x^n.$$

After substitution we get

$$(1 - 2x + x^2)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2\sum_{n=0}^{\infty} c_n x^n = 0,$$

which leads to

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2\sum_{n=2}^{\infty} n(n-1)c_n x^{n-1} + \sum_{n=2}^{\infty} n(n-1)c_n x^n - 2\sum_{n=0}^{\infty} c_n x^n = 0$$

In order to write the series in terms the coefficients of x^n we shift the summation index as

$$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n - 2\sum_{n=1}^{\infty} n(n+1)c_{n+1}x^n + \sum_{n=2}^{\infty} n(n-1)c_n x^n - 2\sum_{n=0}^{\infty} c_n x^n = 0$$

The sum in second and third series can also start from 0 without changing the series. This leads to

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)c_{n+2} - 2n(n+1)c_{n+1} + n(n-1)c_n - 2c_n\right]x^n = 0$$

This can be further simplified as

$$\sum_{n=0}^{\infty}(n+1)\left[(n+2)c_{n+2} - 2nc_{n+1} + (n-2)c_n\right]x^n = 0$$

Equating the coefficients we obtain the recurrence relation

$$(n+2)c_{n+2} - 2nc_{n+1} + (n-2)c_n = 0.$$

Putting $n = 0, 1, 2, \ldots$ we get

$$c_2 = c_0, \quad c_3 = \frac{1}{3}(2c_0 + c_1) =: c, \quad c_4 = c, \quad c_5 = c \ldots$$

Hence the series solution becomes

$$y = c_0 + c_1 x + c_0 x^2 + c\sum_{n=3}^{\infty} x^n.$$

38.1.3 Problem 3

Find the power series solution of the equation $(x^2+1)y'' + xy' - xy = 0$ *in powers of* x *(i.e. about* $x = 0$*).*

Solution: Clearly $x = 0$ is an ordinary point of the given differential equation. Therefore, to find the series solution, we take power series

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots = \sum_{n=0}^{\infty} c_n x^n. \tag{38.3}$$

Differentiating twice in succession, (38.3) gives

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=1}^{\infty} n(n-1) c_n x^{n-2} \tag{38.4}$$

Putting the above value of y, y' and y'' in the given differential equation, we obtain

$$(x^2+1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} n c_n x^{n-1} - x\sum_{n=0}^{\infty} n c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

This leads to

$$\sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n + \sum_{n=1}^{\infty} n c_n x^n - \sum_{n=1}^{\infty} c_{n-1} x^n = 0$$

Finally we have the identity

$$2c_2 + (6c_3 + c_1 - c_0)x + \sum_{n=2}^{\infty} [n(n-1)c_n + (n+2)(n+1)c_{n+2} + n c_n - c_{n-1}]x^n = 0.$$

Equating the constant term and the coefficients of various powers of x, we get

$$c_2 = 0,\ 6c_3 + C_1 - c_0 = 0 \text{ so that } c_3 = (c_0 - c_1)/6$$

and the recurrence relation

$$c_{n+2} = \frac{c_{n-1} - n^2 c_n}{(n+1)(n+2)}, \text{ for all } n \geq 2. \tag{38.5}$$

Putting $n = 2$ in (38.5), $c_4 = (1/12)c_1$, as $c_2 = 0$.

Putting $n = 3$ in (38.5), $c_5 = -\frac{9c_3}{(20)} = -\frac{3}{40}(c_0 - c_1)$

Putting the above values of $c_2, c_3, c_4, c_5, \ldots$ ets. in (38.3), we have

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \ldots\infty$$

$$\Rightarrow \quad y = c_0 + c_1x + (1/6)(c_0 - c_1)x^3 + (1/12)c_1x^4 - (3/40)(c_0 - c_1)x^5 + \ldots\infty$$

This can be rewritten as

$$y = c_0\Big(1 + \frac{1}{6}x^3 - \frac{3}{40}x^5 + \ldots\Big) + c_1\Big(x - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{3}{40}x^5 - \ldots\Big),$$

which is the required solution near $x = 0$, where c_0 and c_1 are arbitrary constants.

38.1.4 Problem 4

Find the power series solution of the initial value problem $xy'' + y' + 2y = 0$, $y(1) = 1$, $y'(1) = 2$ *in powers of* $(x - 1)$.

Solution: Since $x = 1$ is an ordinary point of the given differential equation, we find series solution

$$y = \sum_{n=0}^{\infty} c_n(x-1)^n \quad \Rightarrow \quad y' = \sum_{n=1}^{\infty} nc_n(x-1)^{n-1} \text{ and } y'' = \sum_{n=2}^{\infty} n(n-1)c_n(x-1)^{n-2} \quad (38.6)$$

Substituting y and y' in the given differential equation we obtain

$$[(x-1)+1]\sum_{n=2}^{\infty} n(n-1)c_n(x-1)^{n-2} + \sum_{n=1}^{\infty} nc_n(x-1)^{n-1} + 2\sum_{n=0}^{\infty} c_n(x-1)^n = 0$$

This leads to

$$\sum_{n=2}^{\infty} n(n-1)c_n(x-1)^{n-1} + \sum_{n=2}^{\infty} n(n-1)c_n(x-1)^{n-2} + \sum_{n=1}^{\infty} nc_n(x-1)^{n-1} + 2\sum_{n=0}^{\infty} c_n(x-1)^n = 0$$

Shifting summation index of the first three terms we get

$$\sum_{n=1}^{\infty} n(n+1)c_{n+1}(x-1)^n + \sum_{n=0}^{\infty}[(n+1)(n+2)c_{n+2} + (n+1)c_{n+1} + 2c_n](x-1)^n = 0$$

Equating the coefficients to zero we get

$$2c_2 + c_1 + c_0 = 0 \quad \Rightarrow \quad c_2 = -\frac{c_1 + c_0}{2}$$

$$c_{n+2} = -\frac{(n+1)^2 c_{n+1} + 2c_n}{(n+1)(n+2)}, \text{ for all } n \geq 1$$

Using initial conditions in Equation (38.6) we get $c_0 = 1$ and $c_1 = 2$. Using these values we obtain

$$c_2 = -2, \quad c_3 = \frac{2}{3}, \quad c_4 = -\frac{1}{6}, \quad c_5 = \frac{1}{15}, \ldots$$

Putting these constants in series we get the desired solution as

$$y = 1 + 2(x-1) - 2(x-1)^2 + (2/3)(x-1)^3 - (1/6)(x-1)^4 + (1/15)(x-1)^5 + \ldots$$

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Lesson 39

Series Solutions about a Regular Singular Point

39.1 Introduction

In this lesson we discuss series solution about a singular point. In particular, the power series method discussed in last lessons will be generalized. The generalized power series method is also known as Frobenius method.

Let us consider a simple first order differential equation $2xy' - y = 0$ and try to apply the power series method discussed in the last lessons. Note that $x = 0$ is a singular point. If we plug in

$$y = \sum_{k=0}^{\infty} a_k x^k,$$

into the given differential equation, we obtain

$$0 = 2xy' - y = 2x\left(\sum_{k=1}^{\infty} k a_k x^{k-1}\right) - \left(\sum_{k=0}^{\infty} a_k x^k\right)$$
$$= a_0 + \sum_{k=1}^{\infty} (2k a_k - a_k)\, x^k.$$

First, $a_0 = 0$. Next, the only way to solve $0 = 2ka_k - a_k = (2k-1)\, a_k$ for $k = 1, 2, 3, \ldots$ is for $a_k = 0$ for all k. Therefore we only get the trivial solution $y = 0$. We need a nonzero solution to get the general solution.

39.2 Frobenius Method

Consider the differential equation of the form $y'' + p(x)y' + q(x)y = 0$. Note that $xp(x)$ and $x^2 q(x)$ are analytic at $x = 0$. We try a series solution of the from

$$y = x^r \sum_{n=0}^{\infty} c_n x^n = x^r (c_0 + c_1 x + c_2 x^2 + \ldots), \text{ where } c_0 \neq 0$$

The derivative of y with respect to x are given by

$$y' = \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2}$$

Also, we can write power series corresponding to $xp(x)$ and $x^2q(x)$ as

$$xp(x) = \sum_{n=0}^{\infty} a_n x^n \text{ and } x^2q(x) = \sum_{n=0}^{\infty} b_n x^n$$

The given differential equation can be rewritten as

$$y'' + \frac{xp(x)}{x}y' + \frac{x^2q(x)}{x^2}y = 0$$

Substituting all values of y, y', y'', $xp(x)$ and $x^2q(x)$ series into the above differential equation we get

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} + \sum_{n=0}^{\infty} a_n x^{n-1} \times \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + \sum_{n=0}^{\infty} b_n x^{n-2} \times \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

Multiplying by x^2 we get

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^n \times \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} + \sum_{n=0}^{\infty} b_n x^n \times \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

We can now equate coefficients of various powers of x to zero to form a system of equations involving unknown coefficients c_n. Equating the coefficient of x^r we obtain

$$[r(r-1) + a_0 r + b_0]c_0 = 0$$

Since $c_0 \neq 0$, we obtain

$$r^2 + (a_0 - 1)r + b_0 = 0 \tag{39.1}$$

The above quadratic equation is known as the *indicial equation* of the given differential equation. The general solution of the given differential equation depends on the roots of the indicial equation. There are three possible general cases:

39.2.1 Case I: The indicial equation has two real roots which do not differ by an integer

Let r_1 and r_2 are the roots of the indicial equation. Then the two linearly independent solution will follow from

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} c_n z^n \quad y_2(x) = x^{r_2} \sum_{n=0}^{\infty} \overline{c}_n z^n$$

where $c_0, c_1, \ldots$ are coefficients corresponding to $r = r_1$ and $\overline{c}_0, \overline{c}_1, \ldots$ are coefficients corresponding to $r = r_2$. The general solution will be of the form $y = ay_1 + by_2$, where a and b are arbitrary coefficients.

39.2.2 Case II: The indicial equation has a doubled root

If the indicial equation has a doubled root r, then we find one solution

$$y_1 = x^r \sum_{k=0}^{\infty} a_k x^k,$$

and then obtain another solution by plugging

$$y_2 = x^r \sum_{k=0}^{\infty} b_k x^k + (\ln x) y_1,$$

into the given equation and solving for the constants b_k.

39.2.3 Case III: The indicial equation has two real roots which differ by an integer

If the indicial equation has two real roots such that $r_1 - r_2$ is an integer, then one solution is

$$y_1 = x^{r_1} \sum_{k=0}^{\infty} a_k x^k,$$

and the second linearly independent solution is of the form

$$y_2 = x^{r_2} \sum_{k=0}^{\infty} b_k x^k + C(\ln x) y_1,$$

where we plug y_2 into the given equation and solve for the constants b_k and C.

Remark 1: *Note that the case-I also includes complex numbers because in that case $r_1 - r_2$ will be a complex number which cannot be equal to a real integer.*

Remark 2: *Note that the mai idea is to find at least one Frobenius-type solution. If we are lucky and find two, we are done. If we only get one, we either use the ideas above or the method of variation of parameters to obtain a second solution.*

39.3 Working Rules

Now we summarize the working steps of the Frobenius method:

1. We seek a Frobenius-type solution of the form $y = \sum_{k=0}^{\infty} a_k x^{k+r}$.
2. We plug this y into the given differential equation.
3. The obtained series must be zero. Setting the first coefficient (usually the coefficient of x^r) in the series to zero we obtain the *indicial equation*, which is a quadratic polynomial in r.
4. If the indicial equation has two real roots r_1 and r_2 such that $r_1 - r_2$ is not an integer, then find two linearly independent solutions according to Case-I.
5. If the indicial equation has a doubled root r, or the indicial equation has two real roots such that $r_1 - r_2$ is an integer then follow Case-II or Case-III accordingly.

39.3.1 Example

Find the power series solutions about $x = 0$ of

$$4xy'' + 2y' + y = 0$$

Solution: Clearly, $x = 0$ is a regular singular point. Comparing with $y''+p(x)y'+q(x)y = 0$ we have $xp(x) = 1/2$ and $x^2q(x) = x/4$. We substitute Frobenius series

$$y = x^r \sum_{n=0}^{\infty} c_n x^n \tag{39.2}$$

into the differential equation to get

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2}+\frac{1}{2x}\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1}+\frac{1}{4x}\sum_{n=0}^{\infty}c_n x^{n+r}=0$$

Multiplying by x^2 we obtain

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r}+\frac{1}{2}\sum_{n=0}^{\infty}(n+r)c_n x^{n+r}+\frac{1}{4}\sum_{n=0}^{\infty}c_n x^{n+r+1}=0 \qquad (39.3)$$

Equating coefficients of x^r to zero and noting $c_0 \neq 0$ we obtain indicial equation

$$r(r-1)+\frac{1}{2}r=0$$

which has roots $r = 1/2, 0$. These roots are unequal and do not differ by an integer. To obtain the recurrence relation, we equate to zero the coefficient of x^{n+r} in Equation (39.3) and obtain

$$(n+r)(n+r-1)c_n+\frac{1}{2}(n+r)c_n+\frac{1}{4}c_{n-1}=0$$

Corresponding to $r = 1/2$ we get

$$(4n^2+2n)c_n+c_{n-1}=0 \;\Rightarrow\; c_n=-\frac{c_{n-1}}{2n(2n+1)} \;\Rightarrow\; c_n=-c_0\frac{(-1)^n}{(2n+1)!}$$

Substituting these values in (39.2), we get one solution as

$$y_1=c_0\sqrt{x}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^n=c_0\left(\sqrt{z}-\frac{(\sqrt{z})^3}{3!}+\frac{(\sqrt{z})^5}{5!}+\ldots\right)=\sin\sqrt{z}$$

To obtain the second solution we use $r = 0$ to get

$$(4n^2-2n)c_n+c_{n-1}=0 \;\Rightarrow\; c_n=-\frac{c_{n-1}}{2n(2n-1)} \;\Rightarrow\; c_n=\frac{(-1)^n}{(2n)!}$$

Hence the second solution is

$$y_2=c_0\sum_{n=0}^{\infty}x^n=\cos(\sqrt{z})$$

The general solution is given as

$$y=b\cos(\sqrt{z})+b\cos(\sqrt{z})$$

where a and b are arbitrary constants.

Suggested Readings

Boyce, W.E. and DiPrima, R.C. (2001). Elementary Differential Equations and Boundary Value Problems. Seventh Edition, John Willey & Sons, Inc., New York.

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Lesson 40

Series Solutions about a Regular Singular Point (Cont...)

In this lesson we continue series solution about a singular point. We shall demonstrate the method with some useful differential equations.

40.1 Example Problems

40.1.1 Problem 1

Find one series solution of the differential equation

$$4x^2y'' - 4x^2y' + (1-2x)y = 0,$$

Solution: Note that $x = 0$ is a singular point. Let us try

$$y = x^r \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k x^{k+r},$$

where r is a real number, not necessarily an integer. Again if such a solution exists, it may only exist for positive x. First let us find the derivatives

$$y' = \sum_{k=0}^{\infty} (k+r)\, a_k x^{k+r-1},$$

$$y'' = \sum_{k=0}^{\infty} (k+r)\,(k+r-1)\, a_k x^{k+r-2}.$$

Plugging into our equation we obtain

$$4\sum_{k=0}^{\infty} (k+r)\,(k+r-1)\, a_k x^{k+r} - 4\sum_{k=0}^{\infty} (k+r)\, a_k x^{k+r+1} + (1-2x)\sum_{k=0}^{\infty} a_k x^{k+r} = 0$$

Splitting the last series into two series we get

$$\sum_{k=0}^{\infty} 4(k+r)\,(k+r-1)\, a_k x^{k+r} - \sum_{k=0}^{\infty} 4(k+r)\, a_k x^{k+r+1} + \sum_{k=0}^{\infty} a_k x^{k+r} - 2\sum_{k=0}^{\infty} a_k x^{k+r+1} = 0$$

Re-indexing leads to

$$\sum_{k=0}^{\infty} 4(k+r)\,(k+r-1)\,a_k x^{k+r} - \sum_{k=1}^{\infty} 4(k+r-1)\,a_{k-1} x^{k+r} + \sum_{k=0}^{\infty} a_k x^{k+r} - \sum_{k=1}^{\infty} 2a_{k-1} x^{k+r} = 0$$

Combining different series into one series

$$\big(4r(r-1)+1\big)\,a_0 + \sum_{k=1}^{\infty} \Big(\big(4(k+r)\,(k+r-1)+1\big)\,a_k - \big(4(k+r-1)+2\big)\,a_{k-1} \Big)\, x^{k+r}.$$

The indicial equation is given by

$$4r(r-1)+1 = 0$$

It has a double root at $r = \frac{1}{2}$. All other coefficients of x^{k+r} also have to be zero so

$$\big(4(k+r)\,(k+r-1)+1\big)\,a_k - \big(4(k+r-1)+2\big)\,a_{k-1} = 0.$$

If we plug in $r = \frac{1}{2}$ and solve for a_k, we get

$$a_k = \frac{4(k+\frac{1}{2}-1)+2}{4(k+\frac{1}{2})\,(k+\frac{1}{2}-1)+1}\,a_{k-1} = \frac{1}{k}\,a_{k-1}.$$

Let us set $a_0 = 1$. Then

$$a_1 = \frac{1}{1}a_0 = 1, \qquad a_2 = \frac{1}{2}a_1 = \frac{1}{2},$$
$$a_3 = \frac{1}{3}a_2 = \frac{1}{3\cdot 2}, \qquad a_4 = \frac{1}{4}a_3 = \frac{1}{4\cdot 3\cdot 2}, \cdots$$

In general, we notice that

$$a_k = \frac{1}{k(k-1)(k-2)\cdots 3\cdot 2} = \frac{1}{k!}.$$

In other words,

$$y = \sum_{k=0}^{\infty} a_k x^{k+r} = \sum_{k=0}^{\infty} \frac{1}{k!} x^{k+1/2} = \sqrt{x} \sum_{k=0}^{\infty} \frac{1}{k!} x^k = \sqrt{x} e^x.$$

So we have one solution of the given differential equation. Here we have written the series in terms of elementary functions. However this is not always possible.

40.1.2 Problem 2

Solve the Bessel's equation *of order* p.

$$x^2y'' + xy' + \left(x^2 - p^2\right) y = 0. \tag{40.1}$$

where $2p$ *is not an integer.*

Solution: We take the following generalized power series

$$y = \sum_{m=0}^{\infty} c_m x^{k+m}, \quad c_0 \neq 0. \tag{40.2}$$

which implies

$$y' = \sum_{m=0}^{\infty} c_m (k+m) x^{k+m-1}, \quad y'' = \sum_{m=0}^{\infty} c_m (k+m)(k+m-1) x^{k+m-2}$$

Substitution for y, y', y'' in (40.2) gives

$$x^2 \sum_{m=0}^{\infty} c_m (k+m)(k+m-1) x^{k+m-2} + x \sum_{m=0}^{\infty} c_m (k+m) x^{k+m-1} + (x^2 - n^2) \sum_{m=0}^{\infty} c_m x^{k+m} = 0$$

Combining the first two series we obatin

$$\sum_{m=0}^{\infty} c_m \Big\{ (k+m)(k+m-1) + (k+m) - p^2 \Big\} x^{k+m} + \sum_{m=0}^{\infty} c_m x^{k+m+2} = 0$$

Further simplifications leads to

$$\sum_{m=0}^{\infty} c_m (k+m+p)(k+m-p) x^{k+m} + \sum_{m=0}^{\infty} c_m x^{k+m+2} = 0 \tag{40.3}$$

Equating the smallest power of x to zero, we get the indicial equation as

$$c_0 (k+p)(k-p) = 0, \quad i.e, \quad (k+p)(k-p) = 0, \quad \text{as} \quad c_0 \neq 0.$$

So the roots of indicial equation are $k = p, -p$. Next equating to zero the coefficient of x^{k+1} in (40.3) gives

$$c_1 (k+1+p)(k+1-p) = 0, \quad \text{so that} \quad c_1 = 0 \quad \text{for} \quad k = p \quad \text{and} \quad -p.$$

Finally equating to zero the coefficient of x^{k+m} in (40.3) gives

$$c_m(k+m+p)(k+m-p)+c_{m-2}=0$$

$$\Rightarrow \quad c_m=\frac{1}{(k+m+p)(p-k-m)}c_{m-2}.$$

$$\Rightarrow \quad c_m=\frac{1}{(k+m+p)(p-k-m)}c_{m-2}. \tag{40.4}$$

Putting $m=3,5,7,\ldots$ in (40.4) and using $c_1=0$, we find

$$c_1=c_3=c_5=c_7=\ldots=0.$$

Putting $m=2,4,6,\ldots$ in (40.4), we find

$$c_2=\frac{1}{(k+2+p)(p-k-2)}c_0$$

$$c_4=\frac{1}{(k+4+p)(p-k-4)}c_2=\frac{1}{(k+4+p)(p-k-4)(k+2+p)(p-k-2)}c_0$$

and so on. Putting these values in (40.2) and also replacing c_0 by 1, we get

$$y=\left[1+\frac{x^2}{(k+2+p)(p-k-2)}+\frac{x^4}{(k+4+p)(p-k-4)(k+2+p)(p-k-2)}+\ldots\right].$$

Replacing k by p and $-p$ in the above equation gives

$$y_1=x^p\left[1-\frac{x^2}{4(1+p)}+\ldots\right]=x^p\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{2^{2k}k!(k+p)(k-1+p)\cdots(2+p)(1+p)}$$

$$y_2=x^{-p}\left[1-\frac{x^2}{4(1-p)}+\ldots\right]=x^{-p}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{2^{2k}k!(k-p)(k-1-p)\cdots(2-p)(1-p)}$$

Therefore when $2p$ is not an integer, we have the general solution to Bessel's equation of order p

$$y=c_1y_1(x)+c_2y_2(x),$$

where c_1 and c_2 are arbitrary constants.

Remark: *We define the Bessel functions of the first kind Bessel function of the first kind of order p and $-p$ as*

$$J_p(x) = \frac{1}{2^p\Gamma(1+p)}y_1 = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+p+1)}\left(\frac{x}{2}\right)^{2k+p},$$
$$J_{-p}(x) = \frac{1}{2^{-p}\Gamma(1-p)}y_2 = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k-p+1)}\left(\frac{x}{2}\right)^{2k-p}.$$

As these are constant multiples of the solutions we found above, these are both solutions to Bessel's equation of order p. When p is not an integer, J_p and J_{-p} are linearly independent. When $2p$ is an integer we obtain

$$J_p(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+p)!}\left(\frac{x}{2}\right)^{2k+p}.$$

In this case it turns out that

$$J_p(x) = (-1)^n J_{-p}(x),$$

and so in that case we do not obtain a second linearly independent solution.

40.1.3 Problem 3

Find one series solution of $xy'' + y' + y = 0$.

Solution: The indicial equation is

$$r(r-1) + r = r^2 = 0.$$

This equation has only one root $r = 0$. The recursion equation is

$$(n+r)^2 a_n = -a_{n-1}, \quad n \geq 1.$$

The solution with $a_0 = 1$ is

$$a_n(r) = (-1)^n \frac{1}{(r+1)^2(r+2)^2\cdots(r+n)^2}$$

Setting $r = 0$ gives the solution

$$y_1 = \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{(n!)^2}.$$

Suggested Readings

Boyce, W.E. and DiPrima, R.C. (2001). Elementary Differential Equations and Boundary Value Problems. Seventh Edition, John Willey & Sons, Inc., New York.

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Lesson 41

Introduction

41.1 Introduction to Vector Calculus

We first introduce scalar and vector functions and some basic notation and terminology related to these.

41.1.1 Scalar Function

A scalar function $f(x,y,z)$ is a function defined at each point in a certain domain D in space. It takes real values. It depends on the specific point $P(x,y,z)$ in space, but not on any particular coordinate system which may be used. For every point $(x,y,z) \in D$, f takes a real value. We say the a scalar field f is defined in D. For example, The distance function in the three dimensional space taken as the Euclidean distance between the points $P(x,y,z)$ and $P_0(x_0,y_{0,}z_0)$

$$f(P) = f(x,y,z) = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$$

defines a scalar field.

41.1.2 Vector function

A vector function is defined at each point $P \in D$ in three dimensional space by

$$V = V(P) = v_1 i + v_2 j + v_3 k$$

and we say that a vector field is defined in D. In Cartesian system of coordinates, it can be written as

$$V = v_1(x,y,z)i + v_2(x,y,z)j + v_3(x,y,z)k.$$

An example of a vector field is the velocity field $V(P)$ defined at any point P on a rotating body.

41.1.3 Level surface

Let $f(x,y,z)$ be a single valued continuous scalar function defined at every point $P \in D$. Then an equation of a surface is defined by $f(x,y,z) = c$, a constant. It is called a level surface of the function.

41.1.4 Example : We determine the level surface of the scalar field in space, defined by the following function $f(x,y,z) = x + y + z$.

We find that $f(x,y,z)=c$ gives $x+y+z=c$ which is equation of a plane. For different c they define parallel planes. Therefore, the level surfaces are parallel planes.

41.1.5 Example: Determine the level surface of the scalar field in space, defined by the function $f(x,y,z)=x^2+9y^2+16z^2$.

Note that $f(x,y,z)=c$ gives $x^2+9y^2+16z^2=c$ which defines ellipsoids. So the level surfaces are ellipsoids.

41.2 Parametric Representation of Vector Functions

In this section we introduce the parametric representation of vector functions.

41.2.1 Parametric representation of curves

The parametric representation of a curve C in the two dimensional Cartesian plane is given by $x=x(t),\ y=y(t),\quad a\le t\le b$. Using this the position vector of a point P on the curve C can be written as $r(t)=x(t)\,i+y(t)\,j$.

Therefore, the position vector of a point on a curve defines a vector function. Similarly a three dimensional curve or a space curve or a space curve C can be parameterized as

$$r(t)=x(t)i+y(t)j+z(t)k,\ a\le t\le b$$

41.2.2 Parametric Form of a Straight Line

The parametric form of a line passing through a point with position vector a and with the direction of vector b is given by

$$r(t)=a+tb=(a_1+tb_1)i+(a_2+tb_2)j+(a_3+tb_3)k$$

41.2.3 Parametric Form of a Circle

The parametric form of the circle $x^2+y^2=a^2,$ is defined by

$$r(t)=a\,cost\,i+a\,sint\,j$$

41.2.4 Parametric Form of an Ellipse

The parametric form of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is given by

$$r(t) = a\cos t\, i + b\, sint\, j$$

41.2.5 Parametric Form of a Parabola:

Let us consider the parabola $y^2 = 4ax$. Now take $y = t$ as one parameter and then we can write the parametric form of the parabola as

$$r(t) = \left(\frac{t^2}{4a}\right) i + tj$$

41.2.6 Parametric Representation of Surfaces

We can give parametric representation of surfaces can be done using two parameters. Let $f(x, y, z) = c$ or $g(x, y, z) = 0$ be the equation of a surface. Let an explicit representation of the surface be written as $z = h(x, y)$. Then, if we substitute $u = x, y = v$, the parametric form of the surface can be reduced to

$$r(u, v) = ui + vj + h(u, v)k.$$

41.2.7 Example

The parametric representation of the cylinder $x^2 + y^2 = a^2$ is

$$r(u, v) = a\cos u\, i + a\sin u\, j + v\, k.$$

41.2.8 Example

The parametric representation of the sphere $x^2 + y^2 + z^2 = a^2$ is given by

$$r(u, v) = a\cos u\cos v\, i + a\sin u\cos v\, j + a\sin v\, k,\ 0 \le u \le 2\pi,\ -\pi/2 \le v \le \pi/2$$

41.2.9 Example

The parametric representation of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

is given by

$$r(u, v) = a\cos\ u\cos v\, i + b\sin u\ \cos v\, j + c\sin v\, k$$

41.3 Limit, Continuity and Differentiability of Vector Function

In this section the analytical concepts of the limit, continuity and differentiability of vector function are introduced.

41.3.1 Limit of Vector Function

The vector function $v(t)$ is said to have the limit p as $t \to l$ if $v(t)$ is defined in some neighbourhood of l, except possibly at $t = l$, and

$$\lim_{t \to l} |v(t) - p| = 0$$

We write p. In the Cartesian system, this implies that limits of the component functions $v_1(t), v_2(t)$ and $v_3(t)$ exist as $t \to l$ and

$$\lim_{t \to l} v_1(t) = p_1, \lim_{t \to l} v_2(t) = p_2, \lim_{t \to l} v_3(t) = p_3$$

where $p = p_1\, i + p\, j + p_3 k$.

41.3.2 Continuity

A vector function $v(t)$ is defined to be continuous at $t = l$, if

(i) $v(t)$ is defined in some neighbourhood of l, (ii)$\lim_{t \to l} v(t)$ exists, and (iii)$\lim_{t \to l} v(t) = v(l)$.

In Cartesian system, this implies that $v(t)$is continuous at $t = l$, if and only if the component functions $v_1(t), v_2(t)$ and $v_3(t)$ are continuous at $t = l$.

41.3.3 Differentiability

A vector function $v(t)$ is said to be differentiable at a point, if the limit

$$\lim_{\Delta t \to 0} \frac{v(t+\Delta t) - v(t)}{\Delta t}$$

exists. If the limit exists, then we write it as $v'(t)$ or as $\frac{dv}{dt}$.

In Cartesian system, this implies that the component functions $v_1(t), v_2(t)$ and $v_3(t)$ are differentiable at a point t, and the limits

$$\lim_{\Delta t \to 0} \frac{v_i(t+\Delta t) - v_i(t)}{\Delta t}, i = 1,2,3 \text{ exist.}$$

Therefore, $v'(t) = v_1'(t)i + v_2'(t)j + v_3'(t)k$

Let $v(t) = r(t) = x(t)i + y(t)j + z(t)k$ be the parametric representation of a curve C.

Then $\frac{dr}{dt} = r'(t) = \frac{dx(t)}{dt}i + \frac{dy(t)}{dt}j + \frac{dz(t)}{dt}k.$

41.3.4 Example

Let us consider the represent of the parabola $y = 1 - 2x^2, -1 \leq x \leq 1$ in parametric form. Using this we will find $r'(0)$ and $r'(\frac{\pi}{4})$.

Assume $x = \sin t$. Then $y = 1 - 2sin^2 t = cos2t$, The range of t is $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$. So

$$r(t) = sint\ i + cos2t\ j, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$

Therefore $r'(t) = cost\ i - 2sin2t\ j$,

Further, $r'(0) = i, r'\left(\frac{\pi}{4}\right) = \left(\frac{i}{\sqrt{2}}\right) - 2j$. The tangent at $t = 0$ is parallel to x-axis.

It may be noted that $t = 0$ gives $x = 0,\ y = 1$ which is the vertex of the parabola.

41.3.5 Example

We find the tangent vector to the curve with parametric representation given by

$x = t^3, y = \frac{t+1}{t}, z = t^2 + 1,$ at the point $t = 2$.

We will also find the parametric representation of the tangent vector.

First note that the position vector of a point on the given curve is

$$r(t) = t^3 i + \left(1 + \frac{1}{t}\right)j + (t^2 + 1)k,\ t \neq 0.$$

Therefore the tangent vector is

$$r'(t) = 3t^2\ i - \frac{1}{t^2}j + 2t\ k$$

and $r'(2) = 12i - \frac{1}{4}j + 4k$.

The position vector of the point at which $r'(2)$ is the tangent is $r(2) = 8i + \frac{3}{2}j + 5k$.

Therefore we require the position vector of a point on the line passing through the point whose position vector is $r(2)$ and has the direction of $r'(2)$. Hence, parametric form of the line is given by

$x = 8 + 12t, y = \frac{3}{2} - \frac{t}{4}, z = 5 + 4t$ or $r'(t) = \left(8, \frac{3}{2}, 5\right) + t(12, -\frac{1}{4}, 4)$.

41.3.6 Higher Order Derivatives and Rules of Differentiation

Assuming that the existence of derivatives, we have the following results

$$v''(t) = v_1''(t)i + v_2''(t)j + v_3''(t)k$$

$$(u+v)' = u' + v'$$

$$(f(t)u(t))' = f'(t)u(t) + f(t)u'(t)$$

where $f(t)$ is any real valued scalar function.

$$(u(t).v(t))' = u(t).v'(t) + u'(t).v(t)$$

$$(u(t) \times v(t))' = u(t) \times v'(t) + u'(t) \times v(t)$$

where . and $\times$ represent the dot and cross products, respectively. It must be mentioned that the cross product of two vectors is not commutative.

41.3.7 Example

Find $v'(t)$ in each of the following cases.

$(i)\ v(t) = (\cos t + t^2)(ti + j + 2k)$ $(ii)\ v(t) = (3ti + 5t^2j + 6k).(t^2i - 2tj + tk)$

Solution

$$(\boldsymbol{i})\ v'(t) = (\cos t + t^2)'(ti + j + 2k) + (\cos t + t^2)(ti + j + 2k)'$$
$$= (-\sin t + 2t)(ti + j + 2k) + (\cos t + t^2)(i)$$
$$= (3t^2 + t\sin t + \cos\ t)i + (2t - \sin t)(j + 2k)$$

$$(\boldsymbol{ii})\ v'(t) = (3ti + 5t^2j + 6k)'.(t^2i - 2tj + tk) + (3ti + 5t^2j + 6k).(t^2i - 2tj + tk)'$$
$$= (3i + 10t\,j)(t^2i + 2tj + tk) + (3ti + 5t^2j + 6k).(2ti - 2j + k)$$
$$= 6 - 21t^2$$

41.3.8 Length of a Space Curve

Let the curve C represented in parametric form as $r = r(t), a \le t \le b$. In Cartesian system, we have $r(t) = x(t)i + y(t)j + z(t)k$. Then, the length of the curve is given by

$$l = \int_a^b [\left(x'(t)\right)^2 + \left(y'(t)\right)^2 + \left(z'(t)\right)^2]^{1/2}\, dt = \int_a^b [r'(t).r'(t)]^{1/2}$$

We observe that the integrand is the norm of $r'(t)$, that is

$$||r'(t)|| = [\left(x'(t)\right)^2 + \left(y'(t)\right)^2 + \left(z'(t)\right)^2]^{1/2}$$

Then, we can write

$$l = \int_a^b ||r'(t)||\, dt$$

Sometimes the notation $|r'(t)|$ is also used instead of $||r'(t)||$.

Now, define the real valued function $s(t)$ as

$$s(t) = \int_a^t [\left(x'(\xi)\right)^2 + \left(y'(\xi)\right)^2 + \left(z'(\xi)\right)^2]^{1/2}\, d\xi = \int_a^t ||r'(\xi)||\, d\xi \qquad (41.3.1)$$

Then, $s(t)$ is the arc length of the curve from its initial point $(x(a), y(a), z(a))$ to an arbitrary point $(x(t), y(t), z(t))$ on the curve C. Therefore, $s(t)$ is the length function. Using relation (41.3.1), it is possible to solve for t as a function of s, that is $t = s(t)$. Then the curve C can be parameterised in terms of the arc length s as

$$r(s) = r(t(s)) = x\big(t(s)\big)i + y\big(t(s)\big)j + z(t(s))k$$

41.3.9 Example

We try to find the length of the Helix which is given by

$$r(t) = a\cos t\, i + a\sin t\, j + ct\, k, a > 0, 0 \le t \le 2\pi.$$

First note that we can write

$$x(t) = a\cos t,\ y(t) = a\sin t,\ z(t) = ct.$$

Hence $x'(t) = -a\sin t, y'(t) = a\cos t, z = c.$

Therefore , we have

$$s = \text{arc length} = \int_0^{2\pi} [a^2 sin^2 t + a^2 cos^2 t + c^2]^{1/2} dt = (2\pi)(a^2 + c^2)^{1/2}$$

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Courant, R. and John, F. (1989), Introduction to Calculus and Analysis, Vol. II, Springer-Verlag, New York.

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Lesson 42

Gradient and Directional Derivative

42.1 Gradient of a Scalar Field

Let $f(x,y,z)$ be a real valued function defining a scalar field. To define the gradient of a scalar field, we first introduce a vector operator called del operator denoted by ∇. We define the vector differential operator in two and three dimensions as

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} \quad \text{and} \quad \nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

The gradient of a scalar field $f(x,y,z)$, denoted by ∇f or grad (f) is defined as

$$\nabla \mathrm{f} = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

Note that the del operator ∇ operates on a scalar field and produces a vector field.

42.1. 1 Example

Find the gradient of the following scalar fields

$(i)\ f(x,y) = y^2 - 4xy$ at (1,2),

Solution

$$\nabla \mathrm{f(x,y)} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}\right)(y^2 - 4xy) = -4y\,i + (2y - 4x)j$$

42.1. 2 Example

$\boldsymbol{r} = xi + yj + zk, |\boldsymbol{r}| = r$ and $\hat{r} = \boldsymbol{r}/r$, then show that grad$\left(\frac{1}{r}\right) = -\hat{r}/r^2$.

Solution

$$\text{Grad}\left(\frac{1}{r}\right) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right) = i\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right) + j\left(-\frac{1}{r^2}\frac{\partial r}{\partial y}\right) + k\left(-\frac{1}{r^2}\frac{\partial r}{\partial z}\right) = -\frac{1}{r^2}\left(\frac{x}{r}\,i + \frac{y}{r}j + \frac{z}{r}k\right)$$

$$= -\frac{1}{r^2}\left(\frac{\boldsymbol{r}}{r}\right) = -\frac{\hat{r}}{r^2}$$

where $\hat{r} = (xi + yj + zk)/r$

42.1. 3 Geometrical Representation of the Gradient

Let $f(P) = f(x, y, z)$ be a differentiable scalar field. Let $f(x, y, z) = k$ be a level surface and $P_0(x_0, y_0, z_0)$ be a point on it. There are infinite number of smooth curves on the surface passing through the point P_0. Each of these curves has a tangent at P_0. The totality of these tangent lines form a tangent plane to the surface at a point P_0. A vector normal to this plane at P_0 is called the normal vector to the surface at this point.

Consider now a smooth curve C on the surface passing through a point P on the surface. Let $x = x(t), y = y(t), z = z(t)$ be the parametric representation of the curve C. Any point P on C has the position vector $r(t) = x(t)i + y(t)j + z(t)k$. Since the curve lies on the surface, we have

$$f(x(t), y(t), z(t)) = k$$

Then, $\frac{d}{dt}\ f(x(t), y(t), z(t)) = 0$

By chain rule, we have $\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 0$

or $\left(i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}\right).\left(i\frac{dx}{dt} + j\frac{dy}{dt} + k\frac{dz}{dt}\right) = 0$

or $\nabla f.r'(t) = 0$

Let $\nabla f(P) \neq 0$ and $r'(t) \neq 0$. Now $r'(t)$ is a tangent to C at the point P and lies in the tangent plane to the surface at . Hence $\nabla f(P)$ is orthogonal to every tangent vector at P. Therefore, $\nabla f(P)$ is the vector normal to the surface $f(x, y, z) = k$ at the point P.

42.1. 4 Example

We will find a unit normal vector to the surface $xy^2 + 2yz = 8$ at the point $(3, -2, 1)$.

Let $f(x, y, z) = xy^2 + 2yz = 8$ then

$$\frac{\partial f}{\partial x} = y^2, \frac{\partial f}{\partial y} = 2xy + 2z \text{ and } \frac{\partial f}{\partial z} = 2y$$

Therefore

$$\nabla \mathrm{f} = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} = y^2 i + (2xy + 2z)j + 2yk$$

At $(3, -2, 1)$, we obtain the normal vector as $\nabla \mathrm{f}(3, -2, 1) = 4\mathrm{i} - 10\mathrm{j} - 4\mathrm{k}$. The unit normal vector at $(3, -2, 1)$ is given by

$$\frac{4i - 10j - 4k}{\sqrt{16 + 100 + 16}} = \frac{2i - 5j - 2k}{\sqrt{33}}.$$

42.1. 5 Example

Here we will find the angle between the two surfaces $xlog\ z = y^2 - 1$ and $x^2y = 2 - z$ at the given point (1,1,1).

First note that the angle between two surfaces at a common point is the angle between their normals at that point. Now we have

$$f_1(x,y,z) = xlog\ z - y^2 + 1 = 0, \Delta f_1(x,y,z) = (\log z)i - 2yj + (x/z)k$$

$$\Delta f_1(1,1,1) = -2j + k = n_1$$

$$f_2(x,y,z) = x^2y - 2 + z = 0,\ \Delta f_2(x,y,z) = 2xyi + x^2j + k$$

$$\Delta f_2(1,1,1) = 2i + j + k = n_2$$

Therefore $\cos\theta = \left|\frac{n_1.n_2}{|n_1||n_2|}\right| = \frac{1}{\sqrt{30}}$ or $\theta = cos^{-1}\left(\frac{1}{\sqrt{30}}\right)$.

42.1.6 Properties of Gradient

Let f and g be any two differentiable scalar fields. The gradient satidfies the following algebraic properties,

$$\Delta(f+g) = \Delta f + \Delta g$$

$$\Delta(c_1 f + c_2 g) = c_1\Delta f + c_2\Delta g,\ \text{where } c_1, c_2 \text{ are arbitrary constants}$$

$$\Delta(fg) = f\Delta g + g\Delta f$$

$$\Delta\left(\frac{f}{g}\right) = \frac{g\Delta f - f\Delta g}{g^2}$$

42.2 Directional Derivative

Let $f(P) = f(x,y,z)$ be a differentiable scalar field.

Then $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ denotes the rates of change of f in the direction of x, y and z axis, respectively.

If $f(x,y,z) = k$ is the level surface and P_0 is any point, then $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ at $P_0(x_0,y_0,z_0)$ denote the slopes of the tangent lines in the directions of i,j,k respectively. It is natural to give the definition of derivative in any direction which we call as the directional derivative.

Let $\hat{b} = b_1 i + b_2 j + b_3 k$ be any unit vector. Let P_0 be any point $P_0: a = a_1 i + a_2 j + a_3 k$.

Then, the position vector of any point Q on the line passing through P_0 and in the direction of $\hat{b}$ is given by

$$r = a + t\hat{b} = (a_1 + tb_1)i + (a_2 + tb_2)j + (a_3 + tb_3)k = x(t)i + y(t)j + z(t)k$$

This is, the point $Q(a_1 + tb_1, a_2 + tb_2, a_3 + tb_3)$ is on this line. Now, the vector formthe point P_0 to Q is given by $t\hat{b}$. Since $|\hat{b}|$=1, the distance from P_0 to Q is t. Then

$$\frac{\partial f}{\partial t} = \lim_{t\to 0} \frac{f(Q)-f(P)}{t}$$

if it exists, is called the directional derivative of f at the point P_0 in the direction to $\hat{b}$.

Therefore $\frac{\partial}{\partial t}\ f(x(t), y(t), z(t))$ is rate of change of f with respect to the distance t.

We have

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

where

$\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ are evaluated at $t = 0$.

We write

$$\frac{\partial f}{\partial t} = \left(i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}\right).\left(i\frac{dx}{dt} + j\frac{dy}{dt} + k\frac{dz}{dt}\right) = \nabla f.\frac{dr}{dt}$$

But $\frac{dr}{dt} = \hat{b}$(a unit vector). Therefore, the directional derivative of f in the direction of $\hat{b}$ in given by

$$\text{Directional derivative} = \nabla f.\hat{b} = \text{grad}(f).\hat{b},$$

which is denoted by $D_b(f)$. Note that $\hat{b}$ is a unit vector. If the direction is specified by a vector u, then $\hat{b} = u/|u|$.

42.2.1 Example

We will determine the directional derivative of $f(x, y, z) = xy^2 + 4xyz + z^2$ at the point (1,2,3) in the direction of $3i + 4j - 5k$.

Consider

$$\nabla f = (y^2 + 4yz)i + (2xy + 4xz)j + (4xy + 2z)k.$$

At the point (1,2,3), we have $\nabla f = 28i + 16j + 14k$. The unit vector in the given direction is $\hat{b} = (3i + 4j - 5k)/5\sqrt{2}$.

Therefore

$$D_b(1,2,3) = \frac{1}{5\sqrt{2}}(28i + 16j + 14k).(3i + 4j - 5k) = \frac{78}{5\sqrt{2}}$$

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Lesson 43

Divergence and Curl

43.1 Divergence of a Vector Field

Let $v = v_1(x,y,z)i + v_2(x,y,z)j + v_3(x,y,z)k$ define a vector field.

We define the divergence of vector field as below:

Divergence of v, denoted bi div v , is defined as the scalar

$$\text{div } v = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

Also div $v = \nabla . v = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right).(v_1 i + v_2 j + v_3 k) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$

43.1.1 Example

Here we will find the divergence of the vector field $v = (x^2y^2 - z^3)i + 2xyzj + e^{xyz}k$.

Note that we have

$$\text{div } v = \frac{\partial}{\partial x}(x^2y^2 - z^3) + \frac{\partial}{\partial y}(2xyz) + \frac{\partial}{\partial z}(e^{xyz})$$

$$= 2xy^2 + 2xz + xye^{xyz}$$

43.2 Curl of a Vector Field v

Curl of a vector field v, denoted by curl v, is defined as the vector field

Curl $v = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)i + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)j + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)k$

Curl v can also be written in terms of the gradient operator as

$$\text{Curl } v = \nabla \times v = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

43.2.1 Example

Find the curl of the vector field $v = (x^2y^2 - z^3)i + 2xyzj + e^{xyz}k$

Solution

$$\text{Curl } v = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y^2 - z^3 & 2xyz & e^{xyz} \end{vmatrix}$$

$$= i(xze^{xyz} - 2xy) - j(yze^{xyz} - 3z^2) + k(2yz - 2x^2y)$$

43.2.2 Curl of Gradient

Let f be a differentiable vector field. Then

$$\text{Curl}(\text{grad} f)=0 \text{ or } \nabla \times \nabla f = 0$$

Proof : From the definition, we have

$$\nabla \times \nabla f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= i\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial y \partial z}\right) + j\left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial x \partial z}\right) + k\left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial x \partial y}\right) = 0$$

43.2.3 Divergence of Curl

Let v be a differentiable vector field. Then

$$\text{div}(\text{curl } v)=0 \text{ or } \nabla.(\nabla \times v) = 0$$

proof. Form the definition, we have for $v = v_1 i + v_2 j + v_3 k$

$$\nabla.(\nabla \times f) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right).\left[\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)i + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)j + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)k\right]$$

$$\frac{\partial}{\partial x}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) = 0$$

43.2.4 Example

Prove that $\text{div}(fv) = f\ (\text{div } v) + \text{grad}(f).\ v$, where f is scalar function

Solution

$$\nabla.(fv) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right).(fv_1 i + fv_2 j + fv_3 k) = \frac{\partial}{\partial x}(fv_1) + \frac{\partial}{\partial y}(fv_2) + \frac{\partial}{\partial z}(fv_3)$$

$$= f\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right) + (v_1 i + v_2 j + v_3 k).\left(i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}\right)$$

$$= f(\nabla . v) + v.(\nabla f) = f(\nabla . v) + \nabla f . v$$

43.2.5 Example

If $\boldsymbol{r} = xi + yj + zk, |\boldsymbol{r}| = r$, show that div $(\boldsymbol{r}/r^3)$=0

Solution

$$\Delta.\left(\frac{\boldsymbol{r}}{r^3}\right) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right).\left(i\frac{x}{r^3} + j\frac{y}{r^3} + k\frac{z}{r^3}\right) = \sum \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right)$$

$$= \frac{3}{r^3} - \frac{3}{r^4}\left(\boldsymbol{x}\frac{\partial r}{\partial x} + \boldsymbol{y}\frac{\partial r}{\partial y} + \boldsymbol{z}\frac{\partial r}{\partial z}\right)$$

Since $r^2 = x^2 + y^2 + z^2$

Therefore, $\Delta.\left(\frac{\boldsymbol{r}}{r^3}\right) = \frac{3}{r^3} - \frac{3}{r^3} = \boldsymbol{0}$

43.2.6 Example

Prove the following identities

(i) curl (fv) =(grad$f$)× $v + f$curl v

(ii) div(grad f)=$\nabla^2 f$ where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian operator

(iii) curl(curl$v$)= $\nabla(\nabla . v) - \nabla^2 v$ or grad(div v) = $\nabla \times (\nabla \times v) + \nabla^2 v$.

where f is a scalar function.

Solution

(i) curl $(fv) = \nabla \times (fv) = \nabla \times (fv_1 i + fv_2 j + fv_3 k) = \sum[\frac{\partial}{\partial y}(fv_3) - \frac{\partial}{\partial z}(fv_2)]$

$$= f\left[\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)i + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)j + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)k\right] + [\left(v_3\frac{\partial f}{\partial y} - v_2\frac{\partial f}{\partial z}\right)i + \left(v_1\frac{\partial f}{\partial z} - v_3\frac{\partial f}{\partial x}\right)j + \left(v_2\frac{\partial f}{\partial x} - v_1\frac{\partial f}{\partial y}\right)k]$$

$$= f(\text{curl } v) + \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times (v_1 i + v_2 j + v_3 k)$$

$$= f(\text{curl } v) + (\text{grad } f) \times v$$

(ii) div(grad f)= $\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right).\left(i\frac{\partial f}{\partial x}+j\frac{\partial f}{\partial y}+k\frac{\partial f}{\partial z}\right)=\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}\right)=\nabla^2 f$

(iii) grad(div v)= $\nabla\times(\nabla\times v)=\left(\sum i\frac{\partial}{\partial x}\right)\times\left[\sum i\left(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}\right)\right]$

$$=\sum i\left[\frac{\partial}{\partial y}\left(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x}\right)\right]$$

$$=\sum i\left[\frac{\partial}{\partial x}\left(\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}\right)-\left(\frac{\partial^2 v_1}{\partial x^2}+\frac{\partial^2 v_1}{\partial y^2}+\frac{\partial^2 v_1}{\partial z^2}\right)\right]$$

$$=\left(\sum i\frac{\partial}{\partial x}\right)(\nabla.v)-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)(\sum iv_1)$$

$$=\nabla(\nabla.v)-\nabla^2 v$$

Suggested Readings

Courant, R. and John, F. (1989), Introduction to Calculus and Analysis, Vol. II, Springer-Verlag, New York.

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Lesson 44

Line Integral

44.1 Introduction

Let C be a simple curve. Let the parametric representation of C be written as

$$x = x(t), y = y(t), z = z(t),\ a \le t \le b \qquad (44.1.1)$$

Therefore, the position vector of appoint on the curve C can be written as

$$r(t) = x(t)i + y(t)j + z(t)k, a \le t \le b \qquad (44.1.2)$$

44.2 Line Integral with Respect to Arc Length

Let C be a simple smooth curve whose parametric representation is given as Eqs.(1) and (2). Let $f(x, y, z)$ be continuous on C. Then, we define the line integral f of over C with respect to the arc length s by

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt$$

since

$$ds = \frac{ds}{dt}\,dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

44.2.1 Example

Evaluate $\int_C (x^2 + yz)ds$, where C is the curve defined by $x = 4y, z = 3$ form $(2, \frac{1}{2}, 3)$ to (4,1,3).

Solution

Let $x = t$. Then, $y = t/4$ and $z = 3$. Therefore, the curve C represented by

$x = t, y = \frac{t}{4}, z = 3, 2 \le t \le 3.$

We have $ds = \sqrt{17}/4$.

Hence $\int_C (x^2 + yz)ds = \frac{\sqrt{17}}{4}\int_2^4 \left(t^2 + \frac{3}{4}t\right)dt = \frac{139\sqrt{17}}{24}$.

44.2.2 Line Integral of Vector Fields

Let C be a smooth curve whose parametric representation is given in Eqs. (44.1.1) and (44.1.2). Let

$$v(x,y,z) = v_1(x,y,z)i + v_2(x,y,z)j + v_3(x,y,z)k$$

be a vector field that is continuous on C. Then, the line integral of v over C is defined by

$$\int_C v.dr = \int_C v_1 dx + v_2 dy + v_3 dz$$

$$= \int_C v\big(x(t), y(t), z(t)\big).\frac{dr}{dt}\, dt \qquad (44.2.1)$$

If $v = v_1(x,y,z)i$, then Eq.(44.2.1) reduces to

$$\int_C v.dr = \int_C v_1\, dx = \int_C v_1(x(t), y(t), z(t))\, \frac{dx}{dt} dt$$

Similarly, if $v = v_2(x,y,z)j$ or $v = v_3(x,y,z)k$, we respectively obtained

$$\int_C v.dr = \int_C v_2\, dx = \int_C v_2(x(t), y(t), z(t))\, \frac{dy}{dt} dt$$

and $$\int_C v.dr = \int_C v_3\, dx = \int_C v_3(x(t), y(t), z(t))\, \frac{dy}{dt} dt.$$

44.2.2 Example

Evaluate the line integral of $v = xyi + y^2 j + e^z k$ over the curve C whose parametric representation is given by $x = t^2, y = 2t, 0 \le t \le 1$.

Solution:

The position vector of any point on C is given by $r = t^2 i + 2tj + tk$. We have

$$\int_C v.\frac{dr}{dt}\, dt = \int_0^1 (2t^3 i + 4t^2 j + e^t k).(2ti + 2j + k)dt$$

$$= \int_0^1 (4t^4 + 8t^2 + e^t)dt = \frac{37}{15} + e$$

44.2.3 Example

Evaluate the integral $\int_c (x^2 + yz)dz$, where C is given by $x = t, y = t^2, z = 3t, 1 \le t \le 2$.

Solution:

We have $\int_c (x^2 + yz)dz = 2\int_1^2 (t^2 + 3t^3)\, dt = \frac{163}{4}$

44.3 Line Integral of Scalar Fields

Let C be a smooth curve whose parametric representation is as given in Eqs. (44.1.1) and (44.1.2). Let $f(x,y,z)$, $g(x,y,z)$ and $h(x,y,z)$be scalar fields which are continuous at point over C. Then, we define a line integral as

$$\int_C f(x,y,z)dx + g(x,y,z)dy + h(x,y,z)dz$$

$$= \int_C \left[f\big(x(t), y(t), z(t)\big)\frac{dx}{dt} + g\big(x(t), y(t), z(t)\big)\frac{dy}{dt} + h\big(x(t), y(t), z(t)\big)\frac{dz}{dt} \right] dt$$

If C is closed curve, then we usually write

$$\int_C v.dr = \oint_C v.dr$$

44.3.1 Example

Evaluate $\int_C (x+y)dx - x^2dy + (y+z)dz$, where C is $x^2 = 4y,\ z = x,\ 0 \le t \le 2.$

Solution

First we consider parametric form of C as $x = t, y = \frac{t^2}{4}, z = 2,\ 0 \le t \le 2.$

Therefore,

$$\int_C (x+y)dx - x^2dy + (y+z)dz = \int_0^2 \left[\left(t + \frac{t^2}{4}\right) - t^2\left(\frac{t}{2}\right) + \left(\frac{t^2}{4} + t\right) \right] dt = \frac{10}{3}$$

44.4 Application of Line Integrals

In this section, we consider some physical applications of the concept of line integral.

44.4.1 Work Done By A Force

Let $v(x,y,z) = v_1(x,y,z)i + v_2(x,y,z)j + v_3(x,y,z)k$ be a vector function defined and continuous at every point on C. Then the line integral of tangential component of v along the curve C from a point P to the point Q is given by

$$\int_P^Q v.dr = \int_C v.dr = \int_c v_1dx + v_2dy + v_3dz$$

Let now $v = F$, a variable force acting on a particle which moves along a curve C. Then, the work W done by the force F in displacing the particle from the point P to the point P along the curve C is given by

$$W = \int_P^Q F.dr = \int_{C^*} F.dr$$

where C^* is the part of$C$, whose initial and terminal point are P and Q.

Suppose that $F$is a conservative vector field . Then F can be written as $F = \text{grad}(f)$, where f is a scalar potential(field). Then, the work done

$$W = \int_{C^*} F.dr = \int_{C^*} grad(f).dr$$

$$= \int_{C^*} \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right) = \int_P^Q df = [f(x,y,z)]_P^Q$$

44.4.1 Example

Find the work done by the force $F = -xyi + y^2j + zk$ in moving a particle over the circular path$x^2 + y^2 = 4, z = 0$ form (2,0,0) to (0,2,0).

Solution

The parametric representation of the given curve is $x = 2\cot t, y = 2\sin t, z = 0,\ 0 \le t \le \pi 2$. Therefore, work done W is given by

$$W = \int_C F.dr = \int_C -xydx + y^2dy + zdz$$

$$\int_0^{\pi/2} [-4\sin t\cos t\,(-2\sin t) + 4\,sin^2\,t(2cos)]\,dt = \frac{16}{13}$$

44.4.2 Circulation

A line integral of a vector field v around a simple closed curve C is defined as the circulation of v around C.

Circulation $= \oint_C v.dr = \oint_C v.\frac{dr}{ds}ds = \oint_c v.Tds,$

where T is the tangent vector to C. For example, in fluid mechanics, let v represents the velocity field of a fluid and C be a closed curve in its domain. Then, circulation gives the amount by which the fluid tends to turn the curve rotating or circulating around C. If $\oint_c v.Tds > 0$ then the fluid tends to rotate Cin the anti-clockwise direction, while if $\oint_c v.Tds < 0$, then the fluid tends to rotate C in the clockwise direction perpendicular to$T$ at every point on C, then $\oint_c v.Tds = 0$, that is the curve does not move at all.

44.5 Line Integral Independent of the Path

Let $\phi(x,y,z)$ be a differentiable scalar function. The differential of $\phi(x,y,z)$ is defined as

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = \text{grad}\ \phi.dr$$

Therefore, a differential expression expre $d\phi = f(x,y,z)dx + g(x,y,z)dy + h(x,y,z)dz$ is an exact differential, if there exists a scalar function $\phi(x,y,z)$ such that

$$d\phi = f(x,y,z)dx + g(x,y,z)dy + h(x,y,z)dz.$$

We now present the result on the independence of the path of a line integral

44.5.1 Theorem

Let C be a curve in simply connected domain D in space. Let f, g and h be continuous function having continuous first partial derivatives in D. Then $\int_C f dx + g dy + h dz$ is independent of path C if and only if the integrand is exact differential in D.

44.5.2 Example

Show that $\int_C \frac{x dx + y dy}{\sqrt{x^2+y^2}}$ is independent of path of integration which does not pass through the origin. Find the value of the integral from the point $P(-1,2)$ to the point $Q(2,3)$.

Solution

We have $f(x,y) = \frac{x}{\sqrt{x^2+y^2}}$ and $g(x,y) = \frac{y}{\sqrt{x^2+y^2}}$

Now $\frac{\partial f}{\partial x} = -xy/(x^2+y^2)^{3/2}$ and $\frac{\partial g}{\partial x} = -xy/(x^2+y^2)^{3/2}$

Since $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x}$, the integral is independent of any path of integration which does not pass through the origin. Also, the integrand is an exact differential. Therefore, there exists a function $\phi(x,y)$ such that

$$\frac{\partial \phi}{\partial x} = f(x,y) = \frac{x}{\sqrt{x^2+y^2}} \text{ and } \frac{\partial \phi}{\partial y} = g(x,y) = \frac{y}{\sqrt{x^2+y^2}}$$

Integrating the first equation with respect to x, we get $\phi(x,y) = \sqrt{x^2+y^2} + h(y)$.

Substituting in $\frac{\partial \phi}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}} + \frac{dh}{dy}$ or $\frac{dh}{dy} = 0$ or $h(y) = k$, constant.

Hence $\phi(x,y) = \sqrt{x^2+y^2} + k$

Therefore, $\int_C \frac{x dx + y dy}{\sqrt{x^2+y^2}} = \int_{(-1,2)}^{(2,3)} d(\sqrt{x^2+y^2}) = [\sqrt{x^2+y^2}]_{(-1,2)}^{(2,3)} = \sqrt{13} - \sqrt{5}$

Suggested Readings

Courant, R. and John, F. (1989), Introduction to Calculus and Analysis, Vol. II, Springer-Verlag, New York.

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Lesson 45

Green's Theorem in the Plane

45.1 Introduction

The theorem provides a relationship between a double integral over a region and the line integral over the closed curve C bounding R. Green's theorem is also called the first fundamental theorem of integral vector calculus.

45.2 The Main Result

45.2.1 Theorem: (Green's theorem)

Let C be a piecewise smooth simple closed curve bounding a region R. If $f,\ g,\ \partial f / \partial y, \partial g / \partial x$ are continuous on R, then

$$\int_C f(x,y)dx + g(x,y)dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dxdy$$

The integration being carried in the positive direction (counter clockwise direction) of C.

Proof: We shall prove Green's theorem for a particular case of the region R.

Let the region R be simultaneously expressed in the following forms.

$$R: u_1(x) \le y \le u_2(x),\ a \le x \le b$$

$$R: v_1(x) \le x \le v_2(x),\ c \le y \le d$$

We obtain

$$\iint_R \frac{\partial g}{\partial x} dxdy = \int_c^d \left[\int_{v_1(x)}^{v_2(y)} \frac{\partial g}{\partial x} dx \right] dy = \int_c^d [g(v_2(y), y) - g(v_1(y), y)] dy$$

$$= \int_c^d g(v_2(y), y)dy + \int_d^c g(v_1(y), y)dy = \oint_C g(x,y)dy$$

the integration being carried in the counter clockwise direction.

We obtain

$$\iint_R \frac{\partial f}{\partial x} dxdy = \int_a^b \left[\int_{u_1(x)}^{u_2(y)} \frac{\partial f}{\partial y} dy \right] dy = \int_a^b [f(x, u_2(x)) - f(x, u_1(x))] dx$$

$$= \int_a^b f(x, u_2(x))dx + \int_b^a g(x, u_1(x))dx = -\oint_C f(x,y)dx$$

the integration being carried in the counter clockwise direction. Therefore

$$\int_C f(x,y)dx + g(x,y)dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dxdy .$$

45.2.2 Example : Evaluate

$\int_C (x^2 + y^2)dx + (y + 2x)dy$, where C is the boundary of the region in the first quadrant that is bounded by the curves $y^2 = x$ and $x^2 = y$.

Solution: The curves intersect at (0,0) and (1,1). The bounding curve is C. We have $f(x,y) = x^2 + y^2$ and $g(x,y) = y + 2x$.

Using the Green's theorem, we obtain

$$\int_C (x^2 + y^2)dx + (y + 2x)dy = \iint_R (2 - 2y)dxdy$$
$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (2 - 2y)dydx = \int_0^1 (2y - y^2) \Big|_{x^2}^{\sqrt{x}} dx$$
$$= \int_0^1 (2\sqrt{x} - x - 2x^2 + x^4)dx = 11/30$$

45.2.3 Example: Find the work done by the force $F = (x^2 - y^3)i + (x + y)j$ in moving a particle along the closed path C containing the curves $x + y = 0, x^2 + y^2 = 16$ and $y = x$ in the first and fourth quadrants.

Solution: The work done by the force is given by

$$W = \quad W = \oint_C F.dr = \oint_C (x^2 - y^3)dx + (x + y)dy.$$

The closed path C bounds the region R. Using the Green's theorem, we obtain

$$\int_C (x^2 - y^3)dx + (x + y)dy = \iint_R (1 + 3y^2)dxdy.$$

It is convenient to use polar coordinates to evaluate the integral. The region R is given by

$$R : x = r\cos\theta, y = r\sin\theta, 0 \le r \le 4, -\pi/4 \le \theta \le \pi/4.$$

Therefore,

$$\iint_R (1+3y^2)dxdy = \int_{-\pi/4}^{\pi/4}\int_0^4 (1+3r^2\sin^2\theta)rdrd\theta = \int_{-\pi/4}^{\pi/4}\left[\frac{r^2}{2}+\frac{3}{4}r^4\sin^2\theta\right]_0^4 d\theta$$

$$= \int_{-\pi/4}^{\pi/4}(8+192\sin^2\theta)d\theta = \int_{-\pi/4}^{\pi/4}[8+96(1-\cos 2\theta)]d\theta$$

$$= 2[104\theta - 48\sin 2\theta]_0^{\pi/4} = 52\pi - 96.$$

45.2.4 Example: Verify the Green's theorem for $f(x,y) = e^{-x}\sin y, g(x,y) = e^{-x}\cos y$ and C is the square with vertices at (0,0), (π/2,0), (π/2,π/2), (0,π/2).

Solution: We can write the line integral as

$$\oint_C fdx + gdy = \left[\oint_{C_1} + \oint_{C_2} + \oint_{C_3} + \int_{C_4}\right](fdx+gdy)$$

where C_1, C_2, C_3 and C_4 are the boundary lines. We have along $C_1 : y = 0, 0 \le x \le \pi/2$ and

$$\int_{C_1} e^{-x}(\sin ydx + \cos ydy) = 0,$$

along $C_2 : x = \pi/2, 0 \le y \le \pi/2$ and

$$\int_{C_2} e^{-x}(\sin ydx + \cos ydy) = \int_0^{\pi/2} e^{-\pi/2}\cos ydy = e^{-\pi/2},$$

along $C_3 : y = \pi/2, \pi/2 \le x \le 0$ and

$$\int_{C_3} e^{-x}(\sin ydx + \cos ydy) = \int_{\pi/2}^0 e^{-x}dx = e^{-\pi/2} - 1,$$

along $C_4 : x = 0, \pi/2 \le y \le 0$ and

$$\int_{C_4} e^{-x}(\sin ydx + \cos ydy) = \int_{\pi/2}^0 \cos ydy = -1.$$

Therefore,

$$\oint_C fdx + gdy = \iint_R (-2e^{-x}\cos y)dxdy = \int_0^{\pi/2}\int_0^{\pi/2}(-2e^{-x}\cos ydxdy) = 2(e^{-\pi/2} - 1).$$

45.2.5 Example: Now we use the Green's theorem to show that

$$\int_C \frac{\partial u}{\partial n} ds = \iint_R \nabla^2 u dx dy,$$

where ∇^2 is the Laplace operator $\partial^2 / \partial x^2 + \partial^2 / \partial y^2$ and n is the unit outward normal to C.

Solution:

Let the position vector of a point on C, in terms of the arc length $r(s) = x(s)i + y(s)j.$

Then, the tangent vector to C is given by

$$T = \frac{dr}{ds} = \frac{dx}{ds} i + \frac{dy}{ds} j$$

and the normal vector n is given by (since $n.T = 0$)

$$n = \frac{dy}{ds} i - \frac{dx}{ds} j.$$

Note that n is the unit normal vector. Now

$$\oint_C \frac{\partial u}{\partial n} ds = \oint_C \nabla u.n ds$$

since $\partial u / \partial n$ is the directional derivative of u in the direction of n. Therefore, using Green's theorem, we obtain

$$\oint_C \frac{\partial u}{\partial n} ds = \oint_C \left(\frac{\partial u}{\partial x} \frac{\partial y}{\partial s} - \frac{\partial u}{\partial y} \frac{\partial x}{\partial s} \right) ds = \oint_C \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right)$$

$$= \iint_R \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) dx dy = \iint_R \nabla^2 u dx dy.$$

Suggested Readings

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Lesson 46

Surface Integral

46.1 Introduction

The double and triple integrals are the generalizations of the definite integral $\int_a^b f(x)dx$ to two and three dimensions respectively. The surface area integral is a generalization of the arc length integral

$$\int_a^b \sqrt{1+(y')^2}\,dx.$$

We shall now present a generalization of the line integral $\int_C f(x,y)ds$ to three dimensions. This generalization is called the surface integral.

Let $g(x,y,z)$ be a given function defined in the three dimensional space and let S be surface which is the graph of a function $z = f(x,y)$, or $y = h_1(x,z)$, or $x = h(y,z)$. We assume that (i) $g(x,y,z)$ is continuous at all points on S, (ii) S is smooth and bounded and (iii) the projection R of the surface S on x-y plane, x-z plane, or y-z plane respectively expressed in the forms as assumed in the proof of the Green's theorem. For example, the projection R on the x-y plane can be expressed in the forms

$$R: u_1(x) \le y \le u_2(x),\ a \le x \le b$$

or $R: v_1(x) \le x \le v_2(x),\ c \le y \le d$.

The surface integral can be defined in a similar way as the double integral is defined. Subdivide S into n parts $S_1, S_2, ..., S_n$ of areas $\Delta A_1, \Delta A_2, ..., \Delta A_n$. The projection R of S is therefore partitioned into n rectangles $R_1, R_2, ..., R_n$. We choose an arbitrary point $P_k(x_k, y_k, z_k)$ on each element of the surface area S_k and form the sum $I_n = \sum_{k=1}^{n} g(x_k, y_k, z_k)\Delta A_k$.

Let $n \to \infty$, such that the largest element of the surface area shrinks to a point. This implies that as $n \to \infty$,the length of the longest diagonal of the projected rectangles tends to zero. In the limit as $n \to \infty$, the sequence $\{I_n\}$ has a limiting value which is independent of the way S is subdivided and the choice of P_k on S_k. This limiting value is called the surface integral of $g(x,y,z)$ over S.

That is, we define the surface integral as

$$\iint_S g(x,y,z)dA = \lim_{|d|\to 0}\sum_{k=1}^{n} g(x_k, y_k, z_k)\Delta A_k.$$

where $|d|$ is the length of the longest diagonal of the projected rectangles.

The surface integral can be evaluated in any of the following ways.

(i) Let S be represented in parametric form as $r = r(u,v)$. Then we can write

$$\iint_S g(x,y,z)dA = \iint_{R^*} g[x(u,v), y(u,v), z(u,v)]\,|\, r_u \times r_v \,|\, dudv$$

$$= \iint_{R^*} g[x(u,v), y(u,v), z(u,v)][r_u{}^2 r_v{}^2 - (r_u . r_v)^2]^{1/2}\, dudv$$

where R^* is the region corresponding to S in the u-v plane.

(ii) Let S be represented in the form $z = f(x,y)$. Then we can write

$$\iint_S g(x,y,z)dA = \iint_R g[x, h_1(x,z), z][1 + f_x{}^2 + f_y{}^2]^{1/2}\, dxdy$$

where R is the orthogonal projection of S on the x-y plane.

(iii) Let S be represented in the form $x = h_1(y,z)$. Then we can write

$$\iint_S g(x,y,z)dA = \iint_R g[h(y,z), y, z][1 + (h_1)_x{}^2 + (h_1)^2{}_y]^{1/2}\, dxdz$$

where R is the orthogonal projection of S on the x-z plane.

(iv) Let S be represented in the form $x = h(y,z)$. Then we can write

$$\iint_S g(x,y,z)dA = \iint_R g[h(y,z), y, z][1 + h_y{}^2 + h_z{}^2]^{1/2}\, dydz$$

where R is the orthogonal projection of S on the y-z plane.

If S is piecewise smooth and consists of the surfaces $S_1, S_2, ..., S_k$, then

$$\iint_S g(x,y,z)dA = \iint_{S_1} g(x,y,z)dA + \iint_{S_2} g(x,y,z)dA + ... + \iint_{S_k} g(x,y,z)dA.$$

We now present some of the important applications of the surface integrals.

46.2 Mass of a Surface

Let $\rho(x,y,z)$ denote the density of a surface S at any point or mass per unit surface area. Then, the mass m of the surface is given by

$$m = \iint_S \rho(x, y, z) dA.$$

46.3 Moment of Inertia

Let $\rho(x, y, z)$ denote the density of a surface S at any point. Then, the moment of inertia I of the mass m with respect to a given axis l is defined by the surface integral

$$I = \iint_S \rho(x, y, z) d^2 dA.$$

where d is the distance of the point (x,y,z) from the reference axis l. If the surface is homogeneous, then $\rho(x, y, z)$ = constant and $\rho(x, y, z)$ =m/A, where A is the surface area of S. Then,

$$I = \frac{m}{A} \iint_S d^2 dA$$

46.3.1 Example: Find the mass of the surface of the cone $z = 2 + \sqrt{x^2 + y^2}, 2 \le z \le 7,$ in the first octant, if the density $\rho(x, y, z)$ at any point of the surface is proportional to its distance from the x-y plane.

Solution: The density is given by $\rho(x, y, z) = cz, c$ is constant. We have

$$z = f(x, y) = 2 + \sqrt{x^2 + y^2}, f_x = \frac{x}{\sqrt{x^2 + y^2}}, f_y = \frac{y}{\sqrt{x^2 + y^2}}$$

$$dA = \sqrt{1 + f_x^{\,2} + f_y^{\,2}}\, dxdy = \sqrt{1 + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}} dxdy = \sqrt{2} dxdy.$$

The projection of S on the x-y plane is given by $R : x^2 + y^2 = 25,$ in the first quadrant.

Therefore, mass of the surface is given by

$$m = \iint_S cz dA = \iint_R c[2 + \sqrt{x^2 + y^2}]\sqrt{2} dxdy$$

$$= c\sqrt{2} \iint_R [2 + \sqrt{x^2 + y^2}] dxdy$$

Substituting $x = r\cos\theta, y = r\sin\theta, 0 \le \theta \le \pi / 2,$ we obtain

$$m = c\sqrt{2} \int_0^5 \int_0^{\pi/2} (2 + r) r dr d\theta = c\sqrt{2} \int_0^{\pi/2} \left(r^2 + \frac{r^3}{3} \right)_0^5 d\theta$$

$$= c\sqrt{2} \left(25 + \frac{125}{3} \right) \frac{\pi}{2} = \frac{100\sqrt{2}}{3} \pi c.$$

46.3.2 Example: Evaluate the integral $\iint_S ydA$ where S is the portion of the cylinder $x = 6 - y^2$ in the first octant bounded by the planes $x = 0, y = 0, z = 0$ and $z = 8.$

Solution: The equation of the surface is in the form $x = h(y, z).$ Here $h(y, z) = 6 - y^2$ and $g(x, y, z) = y.$ We have

$h_y = -2y, h_z = 0, (1 + h_y{}^2 + h_z{}^2)^{1/2} = (1 + 4y^2)^{1/2}.$

The projection of S on the y-z plane is the rectangle OABC with sides $y = 0, y = \sqrt{6}, z = 0$ and $z = 8.$ Therefore,

$$\iint_S ydA = \iint_R y(1+4y^2)^{1/2}\,dydz = \int_0^{\sqrt{6}}\int_0^8 y(1+4y^2)^{1/2}\,dydz$$

$$= 8\left[\frac{(1+4y^2)^{3/2}}{8(3/2)}\right]_0^{\sqrt{6}} = \frac{2}{3}[(25)^{3/2} - 1] = \frac{248}{3}.$$

46.3.3 Example: Evaluate the surface integral $\iint_S F.ndA$ where $F = 6zi + 6j + 3yk$ and S is the portion of the plane $2x + 3y + 4z = 12,$ which is in the first octant.

Solution: Let $f(x, y, z) = 2x + 3y + 4z - 12 = 0$ be the surface. Then

grad $f = 2i + 3j + 4k, n = \frac{gradf}{|gradf|} = \frac{1}{\sqrt{29}}(2i + 3j + 4k).$

Consider the projection of S on the x-y plane. The projection of the portion of the plane ABC in the first octant is the rectangle bounded by $x = 0, y = 0$ and $2x + 3y = 12.$ We have

$$dA = \frac{dxdy}{n.k} = \frac{dxdy}{4/\sqrt{29}}.$$

Therefore, $\iint_S F.ndA = \iint_S \frac{1}{\sqrt{29}}(12z + 18 + 12y)dA.$

From the equation of the surface, we get $4z = 12 - 2x - 3y.$ Hence,

$$\iint_S F.ndA = \iint_S \frac{1}{\sqrt{29}}(54-6x+3y)dA = \frac{1}{4}\iint_R (54-6x+3y)dxdy$$

$$= \frac{1}{4}\int_{x=0}^{6}\left[\int_{y=0}^{(12-2x)/3}(54-6x+3y)dy\right]dx = \frac{1}{6}\int_0^6 (360-102x+7x^2)dx$$

$$= \frac{1}{6}\left[360x-51x^2+\frac{7}{3}x^3\right]_0^6 = 138.$$

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Lesson 47

Stokes's Theorem

47.1 Introduction

Let C be a curve in two dimensions which is written in the parametric form $r = r(s)$. Then, the unit tangent vector to C is given by

$$T = \frac{dx}{ds} i + \frac{dy}{ds} j$$

Let v be written in the form $v = gi - fj$.

Then $v.T = (gi - fj).\left(\frac{dx}{ds} i + \frac{dy}{ds} j\right) = g\frac{dx}{ds} - f\frac{dy}{ds}.$

By Green's theorem , we have

$$\oint_C v.dr = \oint_C v.Tds = \int_C gdx - fdy = \iint_R -\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)dxdy = \iint_R (\nabla \times v).kdxdy. \ \backslash$$

This result can be considered as a particular case of the Stokes's theorem. Extension of the Green's theorem to three dimensions can be done under the following generalizations.

(i) The closed curve C enclosing R in the plane $\rightarrow$ the closed curve C bounding an open smooth orientable surface S (open two sided surface).

(ii) The unit normal n to C $\rightarrow$ the unit outward or inward normal n to S.

(iii) Counter clockwise direction of C $\rightarrow$ the direction of C is governed by the direction of the normal n to S. If n is taken as outward normal, then C is oriented as right handed screw and if n is taken as inward normal, then C is oriented as left handed screw.

47.2 The Main Result

We now state the Stokes's theorem.

47.2.1 Theorem (Stokes's Theorem): Let S be a piecewise smooth orient able surface bounded by a piecewise smooth simple closed curve C. Let $v(x, y, z) = v_1(x, y, z)i + v_2(x, y, z)j + v_3(x, y, z)k$ be a vector function which is continuous and has continuous first order partial derivatives in a domain which contains S. If C is traversed in the positive direction, then

$$\oint_C v.dr = \oint_C (v.T)ds = \iint_S (\nabla \times v).ndA$$

where n is the unit normal to S in the direction o orientation of C.

In terms of components of v we have

$$\oint_C [v_1(x,y,z)dx + v_2(x,y,z)dy + v_3(x,y,z)dz] = \iint_S (\nabla \times v).ndA.$$

47.2.2 Remark: As in divergence theorem, the theorem holds if the given surface S can be subdivided into finitely many special surfaces such that each of these surfaces can be described in the required manner.

47.2.3 Remark: To prove the Stokes's theorem, it is not necessary that the equation of the surface should be simultaneously written in the forms $z = f(x,y), y = g(x,z)$ and $x = h(y,z)$. For example, if we take the question of the surface as $z = f(x,y)$ and assume that $f(x,y)$ has continuous second order partial derivatives then the theorem can be easily proved.

47.2.4 Remark: (Physical interpretation of curl)

We know that in rigid body rotation, if v denotes the tangential (linear) velocity of a point on it, then curl v represents the angular velocity of the uniformly rotating body. We also know that a line integral of a vector field v around a simple closed curve C defines the circulation of v around C. For example, if v denotes the velocity of a fluid, then circulation gives the amount by which the fluid tends to turn the curve by rotating or circulating around C. Therefore, circulation (line integral) is closely related to curl of the vector field. To see this, let C_r be a small circle with centre at $P^*(x^*, y^*, z^*)$. Then, by Stokes's theorem, we have

$$\oint_{C_r} v.dr = \iint_{S_r} curlv.ndA$$

where S_r is a small surface whose bounding curve is C_r. Let $P(x,y,z)$ be any arbitrary point on C_r. We approximate $curlv(P) \approx curlv(P^*)$. Then, we have

$$\oint_{C_r} v.dr = \iint_{S_r} [curlv(P^*)].n(P^*)dA = [curlv(P^*).n(P^*)]\iint_{S_r} dA$$

$$= [curlv(P^*).n(P^*)]A_r$$

where A_r is the surface area of S_r. Let the radius r of C_r tend to zero. Then, the approximation $curlv(P) \approx curlv(P^*)$ becomes more accurate and in the limit as $r \to 0$, we get

$$curlv(P^*).n(P^*) = \lim_{r\to 0} \frac{1}{A_r} \oint_{C_r} v.dr.$$

The left hand side of the above equation is the normal component of curl v. The right hand side of equation is circulation of v per unit area. The left hand side is maximum when the circle C_r is positioned such that the normal to surface, $n(P^*)$ points in the same direction as $curlv(P^*)$.

47.2.5 Remark: Stokes's theorem states that the value of the surface integral is same for any surface as long as the boundary curve, bounding the projection R on any coordinate plane, is the same curve C. Hence, in the degenerate case, when S coincides with R, we can take n=k or j or i depending on whether the projection is taken on the x-y plane or x-z plane or y-z plane.

47.2.6 Example: Verify Stokes's theorem for the vector field $v = (3x - y)i - 2yz^2 j - 2y^2 zk$,

where S is the surface of the sphere $x^2 + y^2 + z^2 = 16, z > 0$.

Solution: Consider projection of S on the x-y plane. The projection is the circular region $x^2 + y^2 \le 16, z = 0$ and the bounding curve C is the circle $z = 0, x^2 + y^2 = 16$.

We have

$$\oint_C v.dr = \oint_C (3x - y)dx - 2yz^2 dy - 2y^2 zdz = \oint_C (3x - y)dx$$

since z=0. Setting $x = 4\cos\theta, y = 4\sin\theta$, we obtain

$$\oint_C (3x - y)dx = \int_0^{2\pi} 4(3\cos\theta - \sin\theta)(-4\sin\theta)d\theta = -16\int_0^{2\pi}\left[\frac{3}{2}\sin 2\theta - \frac{1}{2}(1 - \cos 2\theta)\right]d\theta$$

$$= 16\left(\frac{1}{2}\right)2\pi = 16\pi.$$

$$\nabla \times v = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x - y & -2yz^2 & -2y^2 z \end{vmatrix} = i(-4yz + 4yz) - j(0) + k(1) = k$$

Now,

$$n = \frac{2(xi + yj + zk)}{2\sqrt{x^2 + y^2 + z^2}} = \frac{1}{4}(xi + yj + zk), (\nabla \times v).n = \frac{z}{4}.$$

Therefore,

$$\iint_S (\nabla \times v).ndA = \iint_S \frac{z}{4}dA = \iint_R \frac{z}{4}\frac{dxdy}{n.k} = \iint_R \frac{z}{4}\frac{dxdy}{(z/4)} = \iint_R dxdy = 16\pi$$

which is the area of the circular region in the x-y plane. Hence, Stokes's theorem is proved.

47.2.7 Example: Evaluate $\oint_C (2y^3dx + x^3dy + zdz$ where C is the trace of the cone $z = \sqrt{(x^2 + y^2)}$

intersected by the plane x=4 and S is the surface of the cone below z=4.

Solution: We have $v = 2y^3 i + x^3 j + zk$ and

$$curlv = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2y^3 & x^3 & z \end{vmatrix} = i(0) - j(0) + k(3x^2 - 6y^2).$$

If the outward normal to S is taken, then it points downwards. Then, the orientation of C is taken in the clockwise direction. Alternatively, if the inward normal to S is taken, then C is oriented in the counter clockwise direction.

Let $f(x, y, z) = \sqrt{x^2 + y^2} - z = 0$ be taken as the equation of the surface. Then, the normal and unit normal are given by

$$N = \frac{xi + yj}{\sqrt{x^2 + y^2}} - k = \frac{xi + yj - zk}{z} \text{ and } n = \frac{(xi + yj - zk)/z}{\sqrt{(x^2 + y^2 + z^2)/z^2}} = \frac{xi + yj - zk}{\sqrt{2}z} \text{ except at the}$$

origin.

We have $\iint_S (\nabla \times v).ndA = \iint_S -\frac{(3x^2 - 6y^2)}{\sqrt{2}} dA = -\iint_S -\frac{(3x^2 - 6y^2)}{\sqrt{2}} \frac{dxdy}{(-1/\sqrt{2})}$

since $dxdy = (n.k)dA$. Therefore, substituting $x = r\cos\theta, y = r\sin\theta$, we obtain

$$\iint_S (\nabla \times v).ndA = \iint_R (3x^2 - 6y^2)dxdy = \int_{r=0}^{4} \int_{2\pi}^{0} (3\cos^2\theta - 6\sin^2\theta)r^3 drd\theta$$

$$= \frac{3}{2} \int_0^4 \int_{2\pi}^0 [(1 + \cos 2\theta) - 2(1 - \cos 2\theta)]r^3 drd\theta = \frac{3}{2} \int_0^4 \int_{2\pi}^0 (3\cos 2\theta - 1)r^3 drd\theta$$

$$= \frac{3}{2}\left[\frac{r^4}{4}\right]_0^4 \left[\frac{3\sin 2\theta}{2} - \theta\right]_{2\pi}^0 = 192\pi.$$

The bounding curve C is given by $x^2 + y^2 = 16, z = 4$. Now setting $x = 4\cos\theta, y = 4\sin\theta,$

$$\oint_C 2y^3dx + x^3dy + zdz = \oint_C 2y^3dx + x^3dy$$

$$= \int_{2\pi}^{0} 64[2\sin^3\theta(-4\sin\theta) + \cos^3\theta(4\cos\theta)]$$

We obtain

$$= -256\int_0^{2\pi} [\cos^4\theta - 2\sin^4\theta]d\theta = -1024\int_0^{\pi/2} (\cos^4\theta - 2\sin^4\theta$$

$$= -1024\left[\frac{3}{4}.\frac{1}{4}.\frac{\pi}{2} - 2\left(\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}\right)\right] = 192\pi.$$

Hence, the theorem is verified.

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Lesson 48

Divergence Theorem of Gauss

48.1 Introduction

Let C be a curve in two dimensions which is written in the parametric form $r = r(s)$. Then, the unit tangent and unit normal vectors to C are given by

$$T = \frac{dx}{ds}i + \frac{dy}{ds}j, n = \frac{dy}{ds}i - \frac{dx}{ds}j.$$

Then,

$$fdx + gdy = \left(f\frac{dx}{ds} + g\frac{dy}{ds} \right)ds = (gi - fj).\left(\frac{dy}{ds}i - \frac{dx}{ds}j \right)ds = (v.n)ds$$

where $v = gi - fj$. Also

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} \right).(gi - fj) = \nabla.v$$

Hence, Green's theorem can be written in a vector form as

$$\oint_C (v.n)ds = \iint_R (\nabla.v)dxdy$$

The result is a particular case of the Gauss's divergence theorem. Extension of the Greens' theorem to three dimensions can be done under the following generalisations.

(i) A region R in the plane → a three dimensional solid D

(ii) The closed curve C enclosing R in the plane → the closed surface S enclosing the solid D

(iii) The unit outer normal n to C → the unit outer normal n to S.

(iv) A vector field v in the plane → a vector field v in the three dimensional space

(v) The line integral $\oint_C (v.n)ds$ → a surface integral $\iint_S (v.n)dA$

(vi) The double integral $\iint_R \nabla.vdxdy$ → a triple (volume) integral $\iiint_D \nabla.vdV$.

48.2 The Main Result

The above generalizations give the following divergence theorem.

Theorem: (Divergence theorem of Gauss) Let D be a closed and bounded region in the three dimensional space whose boundary is a piecewise smooth surface S that is oriented outward. Let $v(x,y,z) = v_1(x,y,z)i + v_2(x,y,z)j + v_3(x,y,z)k$ be a vector field for which v_1, v_2 and v_3 are continuous first order partial derivatives in some domain containing D. Then,

$$\iint_S (v.n)dA = \iiint_D \nabla.v dV = \iiint_D div(v)dV$$

where n is the outer unit normal vector to S.

Remark: The given domain D can be subdivided into finitely many special regions such that each region can be described in the required manner. In the proof of the divergence theorem, the special region D has a vertical surface. This type of region is not required in the proof. The region may have a vertical surface. For example, the region bounded by a sphere or an ellipsoid has no vertical surface. The divergence theorem holds in all these cases. The divergence theorem also holds for the region D bounded by two closed surfaces.

Remark: In terms of the components of v, divergence theorem can be written as

$$\iint_S v_1 dydz + v_2 dzdx + v_3 dxdy = \iiint_D \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \right) dxdydz$$

or as

$$\iint_S (v_1 \cos\alpha + v_2 \cos\beta + v_3 \cos\gamma)dA = \iiint_D \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \right) dxdydz .$$

Example: Let D be the region bounded by the closed cylinder $x^2 + y^2 = 16, z = 0$ and $z = 4$.

Verify the divergence theorem if $v = 3x^2 i + 6y^2 j + zk$.

Solution: We have $\nabla.v = 6x + 12y + 1$. Therefore,

$$\iiint_D (\nabla.v)dV = \int_{z=0}^{4}\int_{x=-4}^{4}\int_{y=-\sqrt{16-x^2}}^{y=\sqrt{16-x^2}} (6x+12y+1)dydxdz.$$

$$\iiint_D (\nabla.v)dV = \int_{z=0}^{4}\int_{x=-4}^{4}\int_{y=-\sqrt{16-x^2}}^{y=\sqrt{16-x^2}} (6x+12y+1)dydxdz.$$

Since x, y are odd functions, we obtain

$$\iiint_D (\nabla.v)dV = (4)(2)(2)\int_{z=0}^{4}\int_{y=0}^{y=\sqrt{16-x^2}} dydx = 16\int_0^4 \sqrt{16-x^2}\,dx$$

$$= 16\left[\frac{1}{2}x\sqrt{16-x^2} + \frac{16}{2}\sin^{-1}\left(\frac{x}{4}\right)\right]_0^4 = 64\pi.$$

The surface consists of three parts, S_1 (top), S_2 (bottom) and S_3 (vertical),

On $S_1 : z = 4, n = k$

$$\iint_{S_1} (v.n)dA = \iint_{S_1} zdA = 4\iint_{S_1} dA = 4$$ (area of circular region with radius 4)=64π.

On $S_2 : z = 0, n = -k.$

$$\iint_{S_2} (v.n)dA = = \iint_{S_2} -zdA = 0.$$

On $S_3 : x^2 + y^2 = 16, n = \frac{2xi + 2yj}{2\sqrt{x^2+y^2}} = \frac{1}{4}(xi + yj)$

$$\iint_{S_3} (v.n)dA = = \frac{1}{4}\iint_{S_3} (3x^3 + 6y^3)dA.$$

Using the cylindrical coordinates, we write $x = 4\cos\theta, y = 4\sin\theta, dA = 4d\theta dz.$

Therefore,

$$\iint_{S_3} (v.n)dA = \frac{1}{4}\int_{z=0}^{4}\int_{\theta=0}^{2\pi} [192\cos^3\theta + 348\sin^3\theta]4d\theta dz$$

$$= 192\int_0^{2\pi} [(\cos 3\theta + 3\cos\theta) + 2(3\sin\theta - \sin 3\theta)]$$

Hence, $\iint_S (v.n)dA = \iiint_D (\nabla.v)dV.$

Green's Identities (formulas)

Divergence theorem can be used to prove some important identities, called Green's identities which are of use in solving partial differential equations. Let f and g be scalar functions which are continuous and have continuous partial derivatives in some region of the three dimensional space. Let S be a piecewise smooth surface bounding a domain D in this region. Let the functions f and g be such that v=f grad g Then, we have

$$\nabla.(f\nabla g) = f\nabla^2 g + \nabla f.\nabla g$$

By divergence theorem, we obtain

$$\iint_S (v.n)dA = \iint_S f(\nabla g.n)dA = \iiint_D \nabla.(f\nabla g)dV$$

$$\iiint_D (f\nabla^2 g + \nabla f.\nabla g)dV.$$

Now, $\nabla g.n$ is the directional derivative of g in the direction of the unit normal vector n. Therefore, it can be denoted by $\partial g / \partial n.$ We have the Green's first identity as

$$\iint_S f(\nabla g.n)dA = \iint_S f\frac{\partial g}{\partial n}dA = \iiint_D (g\nabla^2 f + \nabla g.\nabla f)dV.$$

Interchanging f and g, we obtain

$$\iint_S g(\nabla f.n)dA = \iint_S f\frac{\partial f}{\partial n}dA = \iiint_D (g\nabla^2 f + \nabla g.\nabla f)dV.$$

Subtracting the two results, we obtain the Green's second identity as

$$\iint_S (f\nabla g - g\nabla f).ndA = \iint_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right)dA = \iiint_D (f\nabla^2 g - g\nabla^2 f)dV.$$

Let f=1. Then, we obtain

$$\iint_S \nabla g.ndA = \iint_S \frac{\partial y}{\partial n} dA = \iiint_D \nabla^2 g dV.$$

If g is a harmonic function, then $\nabla^2 g = 0$ and we have

$$\iint_S \nabla g.ndA = \iint_S \frac{\partial y}{\partial n} dA = 0.$$

This equation gives a very important property of the solutions of Laplace equation, that is of harmonic functions. It states that if $g(x, y, z)$ is a harmonic function, that is, it is a solution of the equation

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2} = 0$$

Then, the integral of the normal derivative of g over any piecewise smooth closed orient able surface is zero.

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