MATRIX

Set of Rows & Column

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix} \rightarrow 1 \text{ row}$$

1×5 → 1 by 5.

one Column

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \rightarrow \text{rows}$$

5×1

Notify [ ] or ( )

$m \times P$ — m → rows, P → Column

$P \times n$ — P → rows, n → Column

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{1n} \\ a_{21} & & & \\ a_{m1} & a_{m2} & a_{m3} & a_{mn} \end{bmatrix}$$

$a_{ij} \Rightarrow$ $i \rightarrow 1, 2, 3 \ldots m$
$j \rightarrow 1, 2, 3 \ldots n$

Various types of Matrices.

① Row matrix → If a matrix has one row & any number of Columns Called row matrix.

ex $\begin{bmatrix} 5 & 6 & 8 \end{bmatrix}$

② Column matrix → a matrix has only one Column & any number of rows Called Column matrix

ex $\begin{bmatrix} 5 \\ 6 \\ 8 \end{bmatrix}$

③ Zero matrix or Null matrix → If all the elements are Zero, that matrix is Called null matrix.

ex $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

④ Square matrix → a matrix in Which the number of rows equal to no. of Columns Such ~~as~~ matrix are Called square matrix.

Ex $\begin{bmatrix} a & b & c \\ p & q & r \\ s & t & v \end{bmatrix}$

(5) Diagonal matrix → A sq matrix is Called diagonal matrix if all its non-diagonal elements are Zero.

Ex $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

(6) Unit or Identity matrix → A square matrix is Called unit matrix if all the diagonal elements are Unity & non diagonal elements are Zero Called Unity or identity matrix.

Ex $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(7) Triangular Matrix (Echelon form)

→ a square matrix, all of whose elements below the leading diagonal are Zero is Called a upper triangular matrix.

→ a Square matrix, all of whose elements above the leading diagonal are Zero is Called lower triangular Matrix.

Ex $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & 6 \end{bmatrix}$ → upper triangular.

$\begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & 6 & 7 \end{bmatrix}$ → lower triangular

Transpose of a Matrix.

When we interchange the rows & the corresponding columns, the new matrix is obtained is called transpose of a matrix. It is denoted by A' or A^T

ex

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 5 \\ 6 & 7 & 8 \end{bmatrix} \qquad A' = \begin{bmatrix} 2 & 1 & 6 \\ 3 & 0 & 7 \\ 4 & 5 & 8 \end{bmatrix}$$

Symmetric Matrix → a sq matrix will be called symmetric matrix, If for all values of i & j

$$a_{ij} = a_{ji}$$

ex.

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \qquad A' = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

$$\boxed{A = A'}$$

Skew-Symmetric Matrix. → a sq matrix is called skew symmetric, if ① $a_{ij} = -a_{ji}$ or $A' = -A$

② all digonal elements are zero.

ex

$$\begin{bmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{bmatrix}$$

Orthogonal Matrix → a square matrix A is called an orthogonal matrix, if the product of the matrix A and the transpose of matrix i.e A' is Identity matrix.

i.e $\boxed{A.A' = I}$

Conjugate of a Matrix :

$$A = \begin{bmatrix} 1+i & 2+3i & 4 \\ 7+2i & -i & 3-2i \end{bmatrix} \qquad \bar{A} = \begin{bmatrix} 1-i & 2+3i & 4 \\ 7-2i & i & 3+2i \end{bmatrix}$$

~~Involuntary matrix →~~

Equal Matrices

Two matrices are said to be equal, if :

1. They are of same order.
2. The elements in the corresponding positions are equal.

Singular Matrix,

The determinant of $|A| = 0$ then A is singular Matrix.

eg $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \Rightarrow |A| = 6 - 6 = 0$.

Multiplication of matrix

If
$$A = \begin{array}{c l} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} & \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \end{array}_{3\times 3} \quad \& \quad B = \begin{array}{c} \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix} \end{array}_{3\times 2}$$

fin AB

$$AB = \begin{bmatrix} 0\times1 + 1\times(-1) + (2\times2) & 0 + 0 + -2 \\ 1 - 2 + 6 & -2 + 0 - 3 \\ 2 - 3 + 8 & -4 + 0 - 4 \end{bmatrix}$$

$$\begin{bmatrix} R_1C_1 & R_1C_2 \\ R_2C_1 & R_2C_2 \\ R_3C_1 & R_3C_2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 3 & -2 \\ 5 & -5 \\ 7 & -8 \end{bmatrix}_{3\times 2}$$

$[A]_{3\times 3}$ $[B]_{3\times 2}$ $\longrightarrow$ Possible

$[B]_{3\times 2}$ $[A]_{3\times 3}$ $\longrightarrow$ Not Possible.

Not same

$$A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

HW

Q. Find the adj & inverse of the following Matrix

$$A = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$\Rightarrow$

$$\begin{array}{c|cccc} \cancel{2} & 5 & 3 & 2 & 5 \\ \hline \cancel{3} & 1 & 2 & 3 & 1 \\ 1 & 2 & 1 & 1 & 2 \\ 2 & 5 & 3 & 2 & 5 \\ \cancel{3} & 1 & 2 & 3 & 1 \end{array}$$

$$\begin{bmatrix} -3 & 1 & 7 \\ -1 & -1 & 5 \\ 5 & 1 & -13 \end{bmatrix}$$

$|A| = 2(-3) - 5(1) + 3(5)$
$= -6 - 5 + 15$
$= -11 + 15$
$= 4$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} -3 & 1 & 7 \\ -1 & -1 & 5 \\ 5 & 1 & -13 \end{bmatrix}$$

$$= \begin{bmatrix} -3/4 & \frac{1}{4} & 7/4 \\ -1/4 & -1/4 & \frac{5}{4} \\ \frac{5}{4} & \frac{1}{4} & \frac{-13}{4} \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 9 & 3 \\ 1 & 4 & 2 \end{bmatrix}$$

$$\begin{array}{c|cccc} \cancel{1} & 1 & 2 & 1 & 1 \\ \hline 1 & 9 & 3 & 1 & 9 \\ 1 & 4 & 2 & 1 & 4 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 9 & 3 & 1 & 9 \end{array}$$

$$\begin{bmatrix} 6 & 6 & -15 \\ 1 & 0 & -1 \\ -5 & -3 & 8 \end{bmatrix}$$

$|B| = 1(6) - 1(-1) + 2(-5)$
$= 6 + 1 - 10 = 7 - 10$
$= -3$

$$B^{-1} = \begin{bmatrix} -2 & -2 & -5 \\ -1/3 & 0 & 1/3 \\ 5/3 & 1 & -8/3 \end{bmatrix}$$

Elementary Transformations

Any one of the following operations on a Matrix is called elementary transformation

① Interchanging any rows (or Columns). This transformation is indicated by R_{ij}, if i^{th} & j^{th} rows are $\boxed{\text{interchanged}}$

② Multiplication of the elements of any row R_i (or Column) by a non zero Scalar quantity K is denoted by $\boxed{K \cdot R_i}$

③ Addition of Constant multiplication of the elements of any row R_j to the corresponding elements of any other row R_i denoted by $\boxed{R_i + K \cdot R_j}$

Gauss-

Finding the inverse of the matrix (only by elementary row transformation).

$$A = I \cdot A$$

$$I = A^{-1} A$$

Calculate the inverse of the Matrix.

1. $\begin{bmatrix} 1 & 2 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A \rightarrow A = I \cdot A$

$R_2 \longrightarrow R_2 - 5R_1$

$\begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix} A \quad \leftarrow I = A^{-1} A$

$R_2 \longrightarrow \frac{R_2}{-3}$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5/3 & -1/3 \end{bmatrix} A$$

$R_1 \longrightarrow R_1 + R_2 \times (-2)$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -7/3 & 2/3 \\ 5/3 & -1/3 \end{bmatrix} A$$

$$\therefore A^{-1} = \begin{bmatrix} -7/3 & 2/3 \\ 5/3 & -1/3 \end{bmatrix}$$

2. $$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \longrightarrow \frac{R_1}{3}$

$$\begin{bmatrix} 1 & -1 & 4/3 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \longrightarrow R_2 + R_1 \times (-2)$

$$\begin{bmatrix} 1 & -1 & 4/3 \\ 0 & -1 & 4/3 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 & 0 \\ -2/3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \longrightarrow \frac{R_2}{-1}$

$$\begin{bmatrix} 1 & -1 & 4/3 \\ 0 & 1 & -4/3 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 & 0 \\ 2/3 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \longrightarrow R_1 + R_2$

ये a wrong hai

real wrong

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4/3 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ \frac{2}{3} & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \rightarrow R_2 + \frac{4}{3} R_3$

$-\frac{4}{3} + \frac{4}{3} \times 1$

$-\frac{1}{3} + \frac{4}{3} = 0$

1 0 0

0 $\frac{1}{3}$

$0 - 2(0)$

$1 - 3(0)$

$0 - 2(-\frac{1}{7})$

$\frac{2}{7}$

$1 - 2(\frac{2}{7})$

$1 - \frac{4}{7}$ $\frac{7-4}{7}$

$0 + 5(-\frac{1}{7})$

$-\frac{5}{7}$

$-\frac{1}{7}$

$-3 + 5(\frac{2}{7})$

$-3 + \frac{10}{7}$

$\frac{-21+10}{7}$

$A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$ find A^{-1}.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow$ $A = I \cdot A$

$I = A^{-1} \cdot A$

$R_1 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 0 & 1 \\ 2 & -3 & -1 & 0 & 1 & 0 \\ 3 & 1 & 2 & 1 & 0 & 0 \end{array}\right]$$

$R_2 \rightarrow R_2 - 2R_1$

$R_3 \rightarrow R_3 - 3R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & -7 & -3 & 0 & 1 & -2 \\ 0 & -5 & -1 & 1 & 0 & -3 \end{array}\right]$$

$R_2 \rightarrow -\frac{1}{7} R_2$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc:ccc} 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & 1 & \frac{3}{7} & 0 & -\frac{1}{7} & \frac{2}{7} \\ 0 & -5 & -1 & 1 & 0 & -3 \end{array}\right]$$

$R_1 \rightarrow R_1 - 2R_2$

$R_3 \rightarrow R_3 + 5R_2$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & \frac{1}{7} & 0 & \frac{2}{7} & \frac{3}{7} \\ 0 & 1 & \frac{3}{7} & 0 & -\frac{1}{7} & \frac{2}{7} \\ 0 & 0 & \frac{8}{7} & 1 & -\frac{5}{7} & -\frac{11}{7} \end{array}\right]$$

$R_3 \rightarrow \frac{7}{8} R_3$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & \frac{1}{7} & 0 & \frac{2}{7} & \frac{3}{7} \\ 0 & 1 & \frac{3}{7} & 0 & -\frac{1}{7} & \frac{2}{7} \\ 0 & 0 & 1 & \frac{7}{8} & -\frac{5}{8} & -\frac{11}{8} \end{array}\right]$$

$R_1 \rightarrow R_1 - \frac{1}{7} R_3$

$R_2 \rightarrow R_2 - \frac{3}{7} R_3$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -\frac{1}{8} & \frac{7}{8} & \frac{5}{8} \\ 0 & 1 & 0 & -\frac{3}{8} & +\frac{1}{8} & \frac{7}{8} \\ 0 & 0 & 1 & \frac{7}{8} & -\frac{5}{8} & -\frac{11}{8} \end{array}\right]$$

Rank of Matrix.

Submatrix → It is a matrix obtained by deleting some rows & some column from a Matrix.

Rank of a Matrix → The rank of a matrix is the order of largest square sub matrix, whose determinant is not equal to zero. Thus, if r is said to be the rank of a matrix A if there exist atleast one square submatrix of A of order are whose determinant is not equal to zero. It is denoted by. r

$r(A)$ or $\rho(A)$

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 6 & 3 \\ 3 & 13 & 14 \end{bmatrix}_{3\times 3}$$

$$|A| = 1\begin{vmatrix} 6 & 3 \\ 13 & 14 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ 3 & 14 \end{vmatrix} + 2\begin{vmatrix} 2 & 6 \\ 3 & 13 \end{vmatrix}$$

$|A| = 1(84-39) + 1(28-9) + 2(26-18)$

$= 1(45) + 1(19) + 2(18)$

$45 + 19 + 36 = 100$ $\Rightarrow |A| \neq 0$

$r(A) = 3$ ∵ rank of the matrix = order of the matrix.

Q: $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix}$

$= B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}_{2\times 2} = |B| = 4-4 = 0.$ $|B| = 0$

$r(A) = 2$

$C = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = |C| = 10-12 = -2$ $|C| \neq 0.$

Normal form of a Matrix

Every non zero matrix of order are can be reduced ~~the form~~ to the form by sequence of elementary transformation where, I_r is the identity matrix of order are which is called normal form of the matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

The other possible normal forms are $[I_r]$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_3 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} I_3 & 0 \end{bmatrix}$$

Rank by Normal form.

By the application of any elementary row & column transformation. A matrix of order m by n. ~~Kind~~ can be reduced to one of the ~~about normo~~ above normal form. then the rank of the given matrix is 'r'

Method

1. 1^{st} element in the first row is 1
2. Convert all elements in its row & column to zero.

(3) 2^{nd} element in the Second row is 1, Convert all elements in its row & Column to zero.

Q Find the rank of matrix by normal form :-

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 1 & 1 & 5 \\ -1 & 0 & -3 \end{bmatrix}$$

= $R_2 \to R_2 + R_1$
$R_3 \to R_3 - R_1$
$R_4 \to R_4 + R_1$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 1 & -2 \end{bmatrix}$$

$C_2 \to C_2 - C_1$
$C_3 \to C_3 - C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 1 & -2 \end{bmatrix}$$

$R_2 \to \frac{1}{2} R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \\ 0 & 1 & -2 \end{bmatrix}$$

$R_4 \to R_4 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & -2 \end{bmatrix}$$

$R_3 \to \frac{1}{4} R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} \qquad R_4 \to R_4 + 2R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_3 \\ 0 \end{bmatrix} \qquad \rho(A) = 3$$

Reduce to normal form & find the rank of the matrix.

$$A = \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$$

$$C_1 \to \frac{C_1}{2}$$

$$A = \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 1 & 3 & 7 & 5 \\ 1 & 5 & 11 & 6 \end{bmatrix}$$

$$R_3 \to R_3 - R_1$$
$$R_4 \to R_4 - R_1$$

$$A = \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & 4 & 4 & 1 \\ 0 & 6 & 8 & 2 \end{bmatrix}$$

$$C_2 \to C_2 + C_1$$
$$C_3 \to C_3 - 3C_1$$
$$C_4 \to C_4 - 4C_1$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 4 & 1 \\ 0 & 4 & 4 & 1 \\ 0 & 6 & 8 & 2 \end{bmatrix} \qquad C_2 \leftrightarrow C_4$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 3 \\ 0 & 1 & 4 & 4 \\ 0 & 2 & 8 & 6 \end{bmatrix} \qquad \cancel{R_3 \to R_3 - R_2}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} R_3 &\to R_3 - R_2 \\ R_4 &\to R_4 - 2R_2 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} C_3 &\to C_3 - 4C_2 \\ C_4 &\to C_4 - 3C_2 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad C_4 \leftrightarrow C_3$$

$$A = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

rank = 3

Q $\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 5 \end{bmatrix}$ $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 3R_1$.

$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 5 \end{bmatrix}$ $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, $C_4 \to C_4 + C_1$

$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_3 \to R_3 - R_2$

$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $C_3 \to C_3 - 2C_2$, $C_4 \to C_4 - 5C_2$

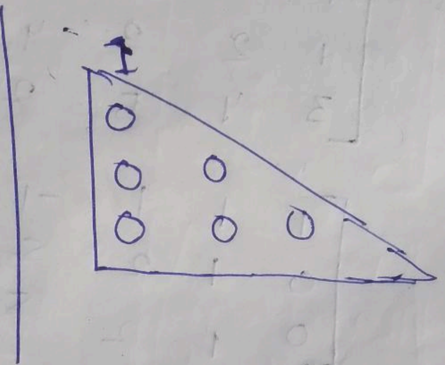
$= \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$ rank = 2

Rank by Echelon form.

Reduced the Given matrix to Echelon form by Using only row-transformation, then the number of non-zeroes rows is the rank of that matrix.

Reduce the matrix to echelon form & find the rank.

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 4 & 6 \end{bmatrix}$$



$= R_2 \to R_2 - R_1$

$R_3 \to R_3 - 2R_1$

$R_4 \to R_4 - 3R_1$

$$= \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$R_4 \to R_4 - R_2 \Rightarrow \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

3 non zeroes row. rank = 3

Q $A = \begin{bmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$

rank = 4

$R_1 \leftrightarrow R_3$

$$A = \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 2 & 2 & 1 \\ 3 & -2 & 0 & -1 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 2 & 2 & 1 \\ 0 & 4 & 9 & -7 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$R_2 \to \frac{R_2}{2}$

$$\begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 1 & \frac{1}{2} \\ 0 & 4 & 9 & -7 \\ 0 & 1 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 5 & -9 \\ 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix}$$

$R_3 \to R_3 - 4R_2$

$R_4 \to R_4 - R_2$.

$R_3 \to \frac{R_3}{5}$.

$$\begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{9}{5} \\ 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix}$$

$R_4 \to R_4 - R_3$.

$$\begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{9}{5} \\ 0 & 0 & 0 & \frac{13}{10} \end{bmatrix}$$

rank = 4

Q. $A = \begin{bmatrix} 3 & -2 & 0 & -1 & -7 \\ 0 & 2 & 2 & 1 & -5 \\ 1 & -2 & -3 & -2 & 1 \\ 0 & 1 & 2 & 1 & -6 \end{bmatrix}$.

$R_3 \leftrightarrow R_1$

$$A = \begin{bmatrix} 1 & -2 & -3 & -2 & 1 \\ 0 & 2 & 2 & 1 & -5 \\ 3 & -2 & 0 & -1 & -7 \\ 0 & 1 & 2 & 1 & -6 \end{bmatrix}$$

$R_3 \to R_3 - 3R_1 \longrightarrow \begin{bmatrix} 1 & -2 & -3 & -2 & 1 \\ 0 & 2 & 2 & 1 & -5 \\ 0 & 4 & 9 & 5 & -10 \\ 0 & 1 & 2 & 1 & -6 \end{bmatrix}$

$R_2 \to \frac{R_2}{2}$;

$$A = \begin{bmatrix} 1 & -2 & -3 & -2 & 1 \\ 0 & 1 & 1 & 1/2 & -5/2 \\ 0 & 4 & 9 & 5 & -10 \\ 0 & 1 & 2 & 1 & -6 \end{bmatrix}$$

$5 - 4 \times \frac{1}{2}$

$-10 - \left(-\frac{1}{2} \times \frac{5}{2}\right)$

$R_3 \to R_3 - 4R_2$

$R_4 \to R_4 - R_2$

$$A = \begin{bmatrix} 1 & -2 & -3 & -2 & 1 \\ 0 & 1 & 1 & 1/2 & -5/2 \\ 0 & 0 & 5 & 3 & 0 \\ 0 & 0 & 1 & -1/2 & -\frac{7}{2} \end{bmatrix}$$

$-\frac{1}{2} - \frac{3}{5}$

$\frac{-5-6}{10}$

$R_3 \to \frac{R_3}{5}$

$$A = \begin{bmatrix} 1 & -2 & -3 & -2 & 1 \\ 0 & 1 & 1 & 1/2 & -5/2 \\ 0 & 0 & 1 & 3/5 & 0 \\ 0 & 0 & 1 & -1/2 & -7/2 \end{bmatrix}$$

$-\frac{7}{2} - 0$

$R_4 \to R_4 - R_3$

$$A = \begin{bmatrix} 1 & -2 & -3 & -2 & 1 \\ 0 & 1 & 1 & 1/2 & -5/2 \\ 0 & 0 & 1 & 3/5 & 0 \\ 0 & 0 & 0 & -\frac{11}{10} & -7/2 \end{bmatrix}$$

rank = 4

Q.1 $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -3 & -6 & -9 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 + 3R_1$
$\to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

rank = 1

Q.2 $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 4 & 6 \end{bmatrix}$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - 2R_1$
$R_4 \to R_4 - 3R_1$
$\to \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

$R_4 \to R_4 - R_2$

$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

rank = 3

Q $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix} \to \begin{matrix} 1 & & \\ 0 & & \\ 0 & 0 & \end{matrix}$

$R_2 \to R_2 - 2R_1$
$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 1 & 3 & 4 & 5 \end{bmatrix} \to$

$R_3 \to R_3 - R_1$
$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix}$

$R_3 \to R_3 + R_1$

$$\begin{bmatrix} -1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

rank = 2

R(A) = 2

9)
$$\begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 2 \\ 3 & -8 & 5 & 2 \\ 5 & -12 & -1 & 6 \end{bmatrix}$$

= $R_2 \to R_2 + 2R_1$
$R_3 \to R_3 + 3R_1$
$R_4 \to R_4 + 5R_1$

$$\rightarrow \begin{bmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & -2 \\ 0 & -2 & 14 & -4 \\ 0 & -2 & 14 & -4 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$
$R_4 \to R_4 - R_3$

$$= \begin{bmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

rank = 2

$\gamma(A) = 2$

$R_4 \to R_4 + R_3$

$$\begin{bmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & -2 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1)
$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$$

2)
$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & -3 \end{bmatrix}$$

3)
$$\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$$

6 − 10

−11 + 15

10 −2 − (−2)

−2 − (−1×2)
−2 + 2
1 + 6
2 + (−4)
2 − 4
−1 − (−4), 14 −11
14 − 14
2 − 6
−4 − (−2×2)
−4 + 4

5 + 9
14 − 14
14 − 14

① $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$ $\longrightarrow$ $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

~~$R(A) = 3$~~ $R_3 - R_2$

$R(A) = 2$. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

② $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & -3 \end{bmatrix}$

$R_2 \to R_2 + R_1$
$R_3 \to R_3 - 2R_1$ $\longrightarrow$ $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & -3 & -5 \end{bmatrix}$

~~$R_2 \to R_2$~~
$R_2 \to \frac{R_2}{2}$.

$-5 + 3 \times \frac{3}{2}$
$-5 + \frac{9}{2}$
$\frac{-10 + 9}{2}$

$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & \frac{3}{2} \\ 0 & -3 & -5 \end{bmatrix}$

$R_3 \to R_3 + 3R_2$ $\longrightarrow$ $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$

$R(A) = 3$

③ $\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$

$C_2 \leftrightarrow C_1$

$\begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 1 & 2 & 3 & 1 \end{bmatrix}$

$R_3 \rightarrow R_3 - R_1$

$\begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 0 & 2 & 1 & 3 \end{bmatrix}$

$R_2 \rightarrow \frac{R_2}{4}$

$\begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 2 & 1 & 3 \end{bmatrix}$

$R_3 \rightarrow R_3 - 2R_2$

$\begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 & -3 \end{bmatrix}$

$R(A) = 3$

Elementary transformation

1. Interchanging :
The interchange of i^{th} rows & j^{th} rows (columns) denoted by

$R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$

2. Scaling the multiplication of the i^{th} row or (column) by non zero scalar k denoted by $R_j \rightarrow kR_j$, $C_i \rightarrow kC_i$

3. Combining : the addition to (or subtraction from the elements of the i^{th} row (or column) of k times the elements of j^{th} row (or column) denoted by

$R_i \rightarrow R_i \pm kR_j$	$R_i \rightarrow R_i \pm kR_j$
$C_i \rightarrow C_i \pm kC_j$	$C_i \rightarrow C_i \pm kC_j$

Linear dependence, Consistency of linear System of equation.

→ A n tuple is set of n similar things. If the place of every members of the set is fixed then it is called ordered set. (1,2) (1,2,3)

→ Any ordered ~~n type of~~ n-tuple of number is called n-vector

(x, y, z)

3-vectors

→ Vectors (Matrices), $x_1, x_2, x_3 \ldots\ldots x_n$ are said to be dependent if

(I) all the vectors are of same order.

(II) n scalar, $\lambda_1 \lambda_2 \ldots\ldots \lambda_n$ (non all zero)

(x, y, z)

$x_1 = (1,2,3)$

$x_2 = (4,5,6)$

$x_1 x_2 \ (1,2,3)_{1\times3}$

exy. such that

$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 \ldots\ldots + \lambda_n x_n = 0$

Otherwise, they are linearly independent.

Q. If in a set of vectors, any vector of the set is combination of the remaining vectors, then the vectors are dependent vectors.

1. Examine the following vectors for linear dependence and find the relation if it exist.

$X_1 = (1,2,4)$, $X_2 = (2,-1,3)$, $X_3 = (0,1,2)$

$X_4 = (-3,7,2)$

$\lambda_1 X_2 + \lambda_2 X_2 + \lambda_3 X_3 + \lambda_4 X_4 = 0$

$\lambda_1(1,2,4) + \lambda_2(2,-1,3) + \lambda_3(0,1,2) + \lambda_4(-3,7,2)$

$\lambda_1 + 2\lambda_2 + 0\lambda_3 - 3\lambda_4 = 0$
$2\lambda_1 - \lambda_2 + \lambda_3 + 7\lambda_4 = 0$
$4\lambda_1 + 3\lambda_2 + 2\lambda_3 + 2\lambda_4 = 0$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2-(-12)$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 4R_1$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 1 \\ 0 & -5 & 2 & 14 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 1 \\ 0 & 0 & 1 & 13 \end{bmatrix}$$

//

$\lambda_1 + 2\lambda_2 - 3\lambda_4 = 0$
$-5\lambda_2 + \lambda_3 + \lambda_4 = 0$
$\lambda_3 + 13\lambda_4 = 0$

let $\lambda_3 = K$
$K + 13\lambda_4 = 0$
$\cancel{1}3\lambda_4 = -K$

$\lambda_4 = \frac{-12}{13}$

Q)

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda_1 + 2\lambda_2 + 0\lambda_3 - 3\lambda_4 = 0 \rightarrow \lambda_1 + 2\lambda_2 - 3\lambda_4 = 0$

$0\lambda_1 - 5\lambda_2 + \lambda_3 + 13\lambda_4 = 0 \rightarrow -5\lambda_2 + \lambda_3 + 13\lambda_4 = 0$

$0\lambda_1 + 0\lambda_2 + 1\lambda_3 + 1\lambda_4 = 0 \rightarrow \lambda_3 + \lambda_4 = 0$

Let $\lambda_4 = t$

$\lambda_3 + t = 0$

$\lambda_3 = -t$

$-5\lambda_2 + \lambda_3 + 13\lambda_4 = 0$

$\cancel{-5\lambda_2 - t + t = 0}$ $\quad -5\lambda_2 - t + 13t = 0$

$\cancel{\lambda_2 = 5}$ $\quad -5\lambda_2 + 12t = 0$

$\lambda_2 = \frac{+12}{+5}t = \frac{12}{5}t$

$\lambda_1 + 2 \times \frac{12}{5}t - 3t = 0$

$\lambda_1 + \frac{24}{5}t - 3t = 0$

$\lambda_1 = 3t - \frac{24}{5}t$

$\lambda_1 = \frac{15t - 24t}{5} = -\frac{9}{5}t$

$\lambda_1 = -\frac{9}{5}t$

$\lambda_2 = \frac{12}{5}t$

$\lambda_3 = -t$

$\lambda_4 = t$

$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4 = 0$

$-\frac{9}{5}t\, x_1 + \frac{12}{5}t\, x_2 - t x_3 + t x_4 = 0$

$-\frac{9}{5}x_1 + \frac{12}{5}x_2 - x_3 + x_4 = 0$

$-9x_1 + 12x_2 - 5x_3 + 5x_4 = 0$ //

Q. Show that row vector of the Matrix

$$\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$ are linearly independent.

Sol^n

$x_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$, $x_2 = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$, $x_3 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0$$

$$\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \lambda_1 + \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} \lambda_2 + \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \lambda_3 = 0$$

$\lambda_1 - \lambda_2 + 0\lambda_3 = 0$ $\rightarrow$ $\lambda_1 - \lambda_2 = 0$

$2\lambda_1 + 3\lambda_2 - 2\lambda_3 = 0$ $\rightarrow$ $2\lambda_1 + 3\lambda_2 - 2\lambda_3 = 0$

$-2\lambda_1 + 0\lambda_2 + 1\lambda_3 = 0$ $\rightarrow$ $-2\lambda_1 + \lambda_3 = 0$

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -2 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$3-(-1\times 2)$
3

$3-(-1\times 2)$
$3+2$

$0+2\times -1$
$1+0\times 2$

$R_2 \rightarrow R_2 - 2R_1$
$R_3 \rightarrow R_3 + 2R_1$

$$\cancel{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 5 & -2 \\ 0 & 3 & -1 \end{bmatrix}} \qquad \begin{bmatrix} 1 & -1 & 0 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

~~$R_3 \rightarrow R_3$~~ $\qquad R_2 \rightarrow \frac{R_2}{5}$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -2/5 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -2/5 \\ 0 & 0 & 1/5 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$1 + \left(\frac{-2}{5}\right)$

$\frac{5-2}{5}$

$1 + \frac{-4}{5}$

$\frac{5-4}{5}$

$1 + \left(\frac{-4}{5}\right)$

$1 + \left(\frac{-4}{5}\right)$

$\frac{5+4}{5}$

$0\lambda_3 + 0\lambda_3 + \frac{9}{5}\lambda_3 = 0$

$\frac{9}{5}\lambda_3 = 0$

$x_3 = 0$

$3/\lambda \, \lambda_3 = 0$

$\frac{1}{5}\lambda_3 = 0$

$\lambda_3 = 0$

$\lambda_3 = 0$

$\lambda_2 + 3/5 \lambda_3 = 0$

$\lambda_2 = 0$

$\lambda_1 - \lambda_2 = 0$

$\lambda_1 = 0$

$\lambda_1 - \lambda_2 + 0\lambda_3 = 0$

$\lambda_2 - \frac{2}{5}\lambda_3 = 0$

$\frac{1}{5}\lambda_3 = 0$

$x_3 = 0$

$\lambda_1 - \lambda_2 + 0\lambda_3 = 0$

$\lambda_2 - 0 = 0$

$\boxed{\lambda_2 = 0}$

$\lambda_2 - \frac{2}{5}\lambda_3 = 0$

$\lambda_1 - \lambda_2 = 0$

$\boxed{\lambda_3 = \frac{1}{5}}$

$\boxed{\lambda_1 = \lambda_2}$

$\lambda_3 = \frac{1}{5}$

$\lambda_3 = 0$

$\lambda_1 = \lambda_2 = 0$

$\frac{1}{5}\lambda_3 = 0$

Linearly dependence and independence of Vectors by rank method.

① If the rank of the matrix of given Vectors is equal to number of Vectors then they are linearly independent.

② If the rank of the matrix is less than number of Vectors, they are linearly dependent.

eg

$X = [1, 2, -3, 4]$
$Y = [3, -1, 2, 1]$
$Z = [1, -5, 8, -7]$

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 2 & 1 \\ 1 & -5 & 8 & -7 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - R_1$.

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 11 & -11 \\ 0 & -7 & 11 & -11 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 11 & -11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R(A) = 2$.
no. of Vectors = 3

$2 < 3$

Here Rank of matrix less than the no. of Vectors So, they are linearly dependent.

$-1-6$
$2+9$
$1-12$
$-7+7$
$-11-(-11)$

Solution of System of Equation.

$$\begin{bmatrix} a_{11} + a_{12} + a_{13} + \cdots + a_{1n} \\ a_{21} + a_{22} + a_{23} + \cdots + a_{2n} \\ \cdots \\ \cdots \\ \cdots \\ a_{m1} + a_{m2} + a_{m3} + a_{m4} + \cdots + a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$$

$$AX = B$$

$$\underbrace{[A\ B]}_{\text{Augmented Matrix}} = C$$

$$\begin{aligned} 2x_1 + 2x_2 + 3x_3 &= 4 \\ 3x_1 + 2x_2 + x_3 &= 5 \\ 4x_1 + 5x_2 + x_3 &= 6 \end{aligned} \Rightarrow \left[\begin{array}{ccc:c} 2 & 2 & 3 & 4 \\ 3 & 2 & 1 & 5 \\ 4 & 5 & 1 & 6 \end{array}\right]$$

$\boxed{AX = B}$ Augmented Matrix

If $r(A) = r[AB] = r = n$. where, n = no. of variables, then the system has unique solution.

If $r(A) = r[A.B] = r < n$, it has infinite solution.

If $r(A) \neq r[AB]$, the system is inconsistent i.e. It has ~~many solution~~. No solution.

Use the list of rank to check the consistency of the following systems of equations & if consistent find the solution.

$$\begin{aligned} x + y + z &= 1 \\ x + 2y + 3z &= 4 \\ x + 3y + 5z &= 7 \\ x + 4y + 7z &= 10. \end{aligned}$$

$$\cancel{[A\ B]} = \cancel{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 4 & 7 \end{bmatrix}}$$

Sol^n

$$[A\ B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 \\ 1 & 4 & 7 & 10 \end{array}\right]$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$R_4 \to R_4 - R_1$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 9 \end{array}\right]$$

$R_3 \to R_3 - 2R_2$

$R_4 \to R_4 - 3R_2$

$$\underbrace{\underbrace{\left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]}_{\rho(A)}}_{\rho[AB]} \Rightarrow \begin{array}{l} \rho(A) = 2 \\ \rho[AB] = 2 \\ n = 3 \end{array}$$

The System has infinite solution.

$x + y + z = 1$

$y + 2z = 3$

Let $z = K$.

$y + 2K = 3$

$y = 3 - 2K$

$x + 3 - 2K + K = 1$

$x + 3 - K = 1$

$x = K - 2$

Q/

$2x + y - 2z = 2$
$x + y + z = 4$
$3x - y + z = 2$
$x + 2y + 2z = 7$

Sol:-

$$\left[\begin{array}{ccc:c} 2 & 1 & -2 & 2 \\ 1 & 1 & 1 & 4 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 2 & 7 \end{array}\right]$$

$R_2 \leftrightarrow R_1$.

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 4 \\ 2 & 1 & -2 & 2 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 2 & 7 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$
$R_4 \to R_4 - R_1$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 4 \\ 0 & -1 & -4 & -6 \\ 0 & -4 & -2 & -10 \\ 0 & 1 & 1 & 3 \end{array}\right]$$

$-2 + 16$

$-10 + 24$
14

$3 + 6$
$= 3$

$R_3 \to R_3 - 4R_2$
$R_4 \to R_4 + R_2$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 4 \\ 0 & -1 & -4 & -6 \\ 0 & 0 & 14 & 14 \\ 0 & 0 & -3 & -3 \end{array}\right]$$

$R_3 \to \frac{R_3}{14}$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 4 \\ 0 & -1 & -4 & -6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -3 & -3 \end{array}\right]$$

$R_4 \to R_4 + 3R_3$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 4 \\ 0 & -1 & -4 & -6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R[A\ B] = 3$

$R[A] = 3$

$n = 3$

It has unique Solution.

$x + y + z = 4$

$-y - 4z = -6$

$\boxed{z = 1}$

$-y - 4 = -6$

$-y = -6 + 4 = -2$

$\boxed{y = 2}$

$x + 2 + 1 = 4$

$\boxed{x = 1}$

~~let z = k~~

~~$k = 1$~~

~~$-y - 4k = -6$~~

~~$-y - 4 = -6$~~

~~$-y = -6 + 4$~~

~~$-y = -2$~~

~~$y = 2$~~

~~$x + y + z = 4$~~

~~$x + 2 + 1 = 4$~~

~~$x = 4 - 3 = 1$~~

~~$x = 1$~~

~~$y = 2$~~

~~$z = 1$~~

~~$k = 1$~~

Investigate for what values of λ & μ the System of equations

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + \lambda z = \mu$$, has

(i) No Solution

(ii) Unique Solution

(iii) Infinite Solution.

Sol^n

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

$C = [A \; B]$

$$\begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & \lambda & : & \mu \end{bmatrix}$$

$R_2 \rightarrow R_1 - R_1$

$R_3 \rightarrow R_3 \rightarrow R_1$

$$\begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 1 & (\lambda-1) & : & (\mu-6) \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & \mu \\ 0 & 0 & (\lambda-3) & : & (\mu-10) \end{bmatrix}$$

~~$R(AB) = 3$~~

~~$R(A) = 3$~~

For no Solution, $\lambda = 3$, $\mu \neq 10$

$R(A) \neq R(C)$

$R(A) = 2$

$R(C) = 3$

For the infinite Solution, $\lambda = 3$, $\mu = 10$

$R(A) = R(C) \cancel{<} n = 3$

$R(A) = 2$

$R(C) = 2$.

For Unique Solution.

$r(A) = r(AB) = n = 3$

$\lambda - 3 \neq 0$

$\lambda \neq 3$, μ can take any Value.

Q $x + 2y + 3z = 4$

$x + 3y + 4z = 5$

$x + 3y + az = b$ has

(i) No Solution

(ii) Unique Solution

(iii) Infinite Solution.

$$= \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 3 & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ b \end{bmatrix}$$

$C = [AB]$

$$= \begin{bmatrix} 1 & 2 & 3 & : & 4 \\ 1 & 3 & 4 & : & 5 \\ 1 & 3 & a & : & b \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc:c} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & (a-3) & (b-4) \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\left[\begin{array}{ccc:c} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & (a-4) & (b-5) \end{array}\right]$$

For no Solution

$r(A) \neq r(AB)$, $a = 4$ but, $b \neq 5$.

for the infinite Solution

$r(A) = r[AB] < n = 3$,

~~a-4=~~ $a = 4$ & $b = 5$

for unique Solution

$r(A) = r[AB] = n$.

$a-4 \neq 0$ $\quad a \neq 4$, b can take any Value.

System of Homogeneous Equation.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 + \dots\dots + a_{1n} = 0$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots\dots\dots + a_{2n} = 0$$

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$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots\dots\dots + a_{mn} = 0$$

It is known as Homogeneous Equations.

Solve

1. $x_1 - x_2 + x_3 = 0$
$x_1 + 2x_2 - x_3 = 0$
$2x_1 + x_2 - 3x_3 = 0$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & -3 \end{bmatrix}$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -2 \\ 0 & 3 & -5 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -2 \\ 0 & 0 & -3 \end{bmatrix}$$

$R(A) = 3$

$\boxed{\gamma(A) = 3 = n}$

Trivial Solution
i.e $x_1 = 0, x_2 = 0,$
$x_3 = 0$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -2 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-3x_3 = 0, \quad x_3 = 0$

$3x_2 - 2x_3 = 0$
$3x_2 = 0, \quad x_2 = 0$

$x_1 = 0$

2. $x - 2y + z - w = 0$
$x + y - 2z + 3w = 0$
$4x + y - 5z + 8w = 0$
$5x - 7y + 2z - w = 0$

$$A = \begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & 1 & -2 & 3 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - 4R_1$
$R_4 \to R_4 - 5R_1$

$$\sim \begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 3 & -3 & 4 \\ 0 & 9 & -9 & 12 \\ 0 & 3 & -3 & 4 \end{bmatrix}$$

$12 - 36$
24

$R_3 \to R_3 - 3R_2$
$R_4 \to R_4 - R_2$

$$\sim \begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 3 & -3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\gamma(A) = 2$
$n = 4$

$\gamma(A) = 2 < n = 4$

It has infinite Solution

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 3 & -3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$x - 2y + z - w = 0$ Let, $w = c_1$
$3y - 3z + 4w = 0$ $\quad z = c_2$

~~$3y = 0$~~

$3y - 3c_2 + 4c_1 = 0$

$y = \dfrac{3c_2 - 4c_1}{3}$

$x - 2\left(\dfrac{3c_2 - 4c_1}{3}\right) + c_2 - c_1 = 0$

$3x - 2[3c_2 - 4c_1] + 3c_2 - 3c_1 = 0$

$3x = 2[3c_2 - 4c_1] - 3c_2 + 3c_1$

$x = \dfrac{6c_2 - 8c_1 - 3c_2 + 3c_1}{3}$

$x = \cancel{6c_3 - 5c_1 - 3c_2}\ \dfrac{3c_2 - 5c_1}{3}$

$x = \frac{1}{3}[3c_2 - 5c_1]$

Eigen Values and Eigen Vectors

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \dots & a_{m} \\ a_{21} & a_{22} & a_{23} \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$AX = Y$

$AX = \lambda X$

$AX = \lambda X$

$AX - \lambda X = 0$

$(A - \lambda) X = 0$

$(A - \lambda I) X = 0$

$(A - \lambda I)$ ↓ Eigen Values

X → eigen Vector.

Find eigen values of

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

~~$|A - \lambda I|$~~ $A - \lambda I = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix}$$

$|A - \lambda I|$

$= \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} \Rightarrow (5-\lambda)(2-\lambda) - (1 \times 4) = 0$

$= 10 - 5\lambda - 2\lambda + \lambda^2 - 4 = 0$

$= 6 - 7\lambda + \lambda^2 = 0$

$\lambda^2 - 7\lambda + 6 = 0$

Characteristic equation

$\lambda^2 - 6\lambda - \lambda + 6 = 0$

$\lambda(\lambda - 6) - 1(\lambda - 6) = 0$

$\lambda = 1$ & $\lambda = 6$

$$A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$$

$A - \lambda I = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \Rightarrow \begin{bmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{bmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{vmatrix}$$

$$= (8-\lambda)(2-\lambda) - (-8)$$

$$16 - 8\lambda - 2\lambda + \lambda^2 + 8 = 0$$

$$24 - 10\lambda + \lambda^2 = 0$$

$$\lambda^2 - 10\lambda + 24 = 0$$

$$\lambda^2 - 6\lambda - 4\lambda + 24 = 0$$

$$\lambda(\lambda-6) - 4(\lambda-6) = 0$$

$$\boxed{\lambda = 6 \ \& \ \lambda = 4}$$

Eigen Values and Eigen Vectors for three

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix}$$

$$2-\lambda \begin{vmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{vmatrix} - 2 \begin{vmatrix} 1 & 1 \\ 1 & 2-\lambda \end{vmatrix} + 1 \begin{vmatrix} 1 & 3-\lambda \\ 1 & 2 \end{vmatrix}$$

$\Rightarrow 2-\lambda\left[(3-\lambda)(2-\lambda)-2\right] - 2\left[\cancel{2-\lambda-1}\right] + 1\left[2-(3-\lambda)\right]$

$\Rightarrow 2-\lambda\left[6-3\lambda-2\lambda+\lambda^2-2\right] -2\left[1-\lambda\right] + 1\left[2-3+\lambda)\right]$

$= 2-\lambda\left[4-5\lambda+\lambda^2\right] -2 + 2\lambda -1 + \lambda$

$= 8-10\lambda+2\lambda^2-4\lambda+5\lambda^2-\lambda^3+3\lambda-3$

$= 5-11\lambda+7\lambda^2-\lambda^3 = 0$

$= \lambda^3-7\lambda^2+11\lambda-5=0$ $\qquad \lambda-1=0$, $x=1$

$\lambda-1=0$

$\lambda=1$

$1-7+11-5=0$

$12-12=0$

$$\begin{array}{r|l} \lambda-1 & \lambda^3-7\lambda^2+11\lambda-5 \quad (\lambda^2-6\lambda+5 \\ & \lambda^3-\lambda^2 \\ & -\quad + \\ \hline & -6\lambda^2+11\lambda-5 \\ & -6\lambda^2+6\lambda \\ & +\quad - \\ \hline & 5\lambda-5 \\ & 5\lambda-5 \\ \hline & 0 \end{array}$$

$(\lambda-1)(\lambda^2-6\lambda+5)=0$

$(\lambda-1)(\lambda^2-5\lambda-\lambda+5)=0$

$(\lambda-1)\left[(\lambda-5)-1(\lambda-5)\right]=0$

$$\boxed{\lambda=1,\ \lambda=5,\ \lambda=1}$$

Cayley Hamilton Theorem

$A \rightarrow$ Square matrix

C.H Theorem

$|A - \lambda I| = 0$

$A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$

Satisfies its own Characteristic equation

To find A^{-1}, if exist

$A^{-1} = \frac{1}{|A|}(\text{adj } A)$, $|A| \neq 0$

A non Singular / inverse.

Q Find the inverse of a matrix of a matrix A, given by

$A = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$, If exist, Using Cayley's hamilton theorem.

Soln $(A - \lambda I) = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$

$= \begin{bmatrix} 4-\lambda & 3 & 1 \\ 2 & 1-\lambda & 2 \\ 1 & 2 & 3-\lambda \end{bmatrix}$

$$(A-\lambda I) = \begin{bmatrix} 4-\lambda & 3 & 1 \\ 2 & 1-\lambda & 2 \\ 1 & 2 & 3-\lambda \end{bmatrix}$$

$$|A-\lambda I| = 4-\lambda \begin{vmatrix} 1-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} - 3 \begin{vmatrix} 2 & 2 \\ 1 & 3-\lambda \end{vmatrix} + 1 \begin{vmatrix} 2 & 1-\lambda \\ 1 & 2 \end{vmatrix}$$

$= 4-\lambda[(1-\lambda)(3-\lambda)-4] - 3[6-2\lambda-2] + 1[4-1+\lambda]$

$= 4-\lambda[3-\lambda-3\lambda+\lambda^2-4] - 3[4-2\lambda] + 1[3+\lambda]$

$= 4-\lambda[\lambda^2-4\lambda-1] - 12 + 6\lambda + 3 + \lambda$

$= 4\lambda^2 - 16\lambda - 4 - \lambda^3 + 4\lambda^2 + \lambda - 9 + 7\lambda$

$= -\lambda^3 + 8\lambda^2 - 8\lambda - 13 = 0$

$= \lambda^3 - 8\lambda^2 + 8\lambda + 13 = 0$

$A^3 - 8A^2 + 8A + 13I = 0I$

Premultiplies of A^{-1}

Using Cayley hamilton theorem :

$A^{-1}A^3 - 8A^{-1}A^2 + 8A^{-1}A + 13A^{-1}I = 0I$

$A^2 - 8A + 8I + 13A^{-1} = 0$

$A^2 - 8A + 8I = -13A^{-1}$

$-13A^{-1} = A^2 - 8A + 8I$

$A^{-1} = \frac{-1}{13}[A^2 - 8A + 8I]$

Now, find A^2

$$A^2 = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 16+6+1 & 12+3+2 & 4+6+3 \\ 8+2+2 & 6+1+4 & 2+2+6 \\ 4+4+3 & 3+2+6 & 1+4+9 \end{bmatrix} = \begin{bmatrix} 23 & 17 & 13 \\ 12 & 11 & 10 \\ 11 & 11 & 14 \end{bmatrix}$$

$$A^{-1} = -\frac{1}{13}\left[\begin{bmatrix} 23 & 17 & 13 \\ 12 & 11 & 10 \\ 11 & 11 & 14 \end{bmatrix} - 8\begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]$$

$$= -\frac{1}{13}\begin{bmatrix} 23-32+8 & 17-24 & 13-8 \\ 12-16 & 11-8+8 & 10-16 \\ 11-8 & 11-16 & 14-24+8 \end{bmatrix}$$

$$= -\frac{1}{13}\begin{bmatrix} -1 & -7 & 5 \\ -4 & 11 & -6 \\ 3 & -5 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{13} & \frac{7}{13} & \frac{-5}{13} \\ \frac{4}{13} & \frac{-11}{13} & \frac{6}{13} \\ \frac{-3}{13} & \frac{5}{13} & \frac{2}{13} \end{bmatrix}$$

Verification

$$AA^{-1} = I$$

$$\begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} \frac{1}{13} & \frac{7}{13} & \frac{-5}{13} \\ \frac{4}{13} & \frac{-11}{13} & \frac{6}{13} \\ \frac{-3}{13} & \frac{5}{13} & \frac{2}{13} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$4\times\frac{1}{13} + 3\times\frac{4}{13} - \frac{3}{13} = \frac{4}{13} + \frac{12}{13} - \frac{3}{13} = \frac{16}{13} - \frac{3}{13} = \frac{13}{13} = 1$$

$$4\times\frac{7}{13} - 3\times\frac{11}{13} + \frac{5}{13} = \frac{28}{13} - \frac{33}{13} + \frac{5}{13} = \frac{33}{13} - \frac{33}{13} = 0$$

$$4\times-\frac{5}{13} + 3\times\frac{6}{13} + 1\times\frac{2}{13} = -\frac{20}{13} + \frac{18}{13} + \frac{2}{13} = 0$$

$$2\times\frac{1}{13} + 1\times\frac{4}{13} - 2\times\frac{3}{13} = \frac{2}{13} + \frac{4}{13} - \frac{6}{13} = 0$$

$$2\times\frac{7}{13} - \frac{11}{13} + \frac{10}{13} = \frac{14}{13} - \frac{11}{13} + \frac{10}{13} = \frac{24}{13} - \frac{11}{13} = \frac{13}{13} = 1$$

Q. $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$

$$(A - \lambda I) = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & -1 & 2 \\ 0 & 2-\lambda & -1 \\ 0 & 0 & 3-\lambda \end{bmatrix}$$

$$|A - \lambda I| = 1-\lambda \begin{vmatrix} 2-\lambda & -1 \\ 0 & 3-\lambda \end{vmatrix} + 1 \begin{vmatrix} 0 & -1 \\ 0 & 3-\lambda \end{vmatrix} + 2 \begin{vmatrix} 0 & 2-\lambda \\ 0 & 0 \end{vmatrix}$$

$$= 1-\lambda[(2-\lambda)(3-\lambda)]$$

$$1-\lambda\ [6-2\lambda-3\lambda+\lambda^2]$$

$$= (1-\lambda)(\lambda^2-5\lambda+6)$$

$$= \lambda^2-5\lambda+6-\lambda^3+5\lambda^2-6\lambda=0$$

$$= -\lambda^3+6\lambda^2-11\lambda+6=0$$

$$\lambda^3-6\lambda^2+11\lambda-6=0$$

$\lambda-1$ is a factor.

$$\lambda-1 \overline{)\ \lambda^3-6\lambda^2+11\lambda-6\ } \ (\lambda^2-5\lambda+6$$

$$\cancel{\lambda^3} - \lambda^2$$

$$-5\lambda^2+11\lambda-6$$

$$-5\lambda^2+5\lambda$$

$$6\lambda-6$$

$$6\lambda-6$$

$$0$$

$(\lambda-1)[\lambda^2-5\lambda+6]$

$(\lambda-1)(\lambda^2-3\lambda-2\lambda+6)$

$(\lambda-1)(\lambda-3)(\lambda-2)$

$\lambda = 1, 3, 2$

$A^3 - 6A^2 + 11A - 6I = 0$

$\cancel{A^{-1}A^3 - 6A^2 + 11I}$

. Premultiplies of A^{-1}

$A^{-1}A^3 - 6A^{-1}A^2 + 11A^{-1}A - 6IA^{-1} = 0A^{-1}$

$A^2 - 6A + 11I - 6A^{-1} = 0$

$6A^{-1} = A^2 - 6A + 11I.$

$$A^2 = \begin{bmatrix} 1 & -3 & 9 \\ 0 & 4 & -5 \\ 0 & 0 & 9 \end{bmatrix},$$

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -3 & 9 \\ 0 & 4 & -5 \\ 0 & 0 & 9 \end{bmatrix} - 6\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{1}{6}\begin{bmatrix} 1-6+11 & -3+6 & 9-12 \\ 0-0+0 & 4-12+11 & -5+6 \\ 0 & 0 & 9-18+11 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{6} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

Verification

$AA^{-1} = I.$

$\lambda = 1, 2, 3$

$\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 3$

Eigen Values Correspondly to λ_1

$(A-\lambda I)X_1 = 0$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix} - 1\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$0x_1 - 1x_2 + 2x_3 = 0$

$0x_1 + x_2 - x_3 = 0$

$0x_1 + 0x_2 + 2x_3 = 0$

~~Let $x_3 = k$~~

$$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = K.$$

$a_1x + b_1y + c_1z = 0$

$a_2x + b_2y + c_2z = 0$

$a_3x + b_3y + c_3z = 0$

Apply Cramer's rule

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{a_2c_1 - a_1c_2} = \frac{z}{a_1b_2 - a_2b_1} = k$$

$\frac{x}{\alpha_1} = K_1, \quad \frac{y}{\alpha_2} = k, \quad \frac{z}{\alpha_3} = K_3$

$x = k_1\alpha_1$, $y = k_2\alpha_2$, $z = k_3\alpha_3$

Some important Properties of eigen Values :

① Any square matrix A & B its transpose A' have the same eigen values.

② The sum of the eigen values of a matrix is equal to the trace of the matrix.

③ The Product of the eigen. Values of a matrix A

④ If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigen values of A, then the eigen values of

(i) kA are $k\lambda_1, k\lambda_2, \ldots k\lambda_n$.

(ii) A^m are $\lambda_1^m, \lambda_2^m \ldots \lambda_n$

(iii) A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3} \ldots \frac{1}{\lambda_n}$.

Q. $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$

find the eigen values of A^{25}

$= (A - \lambda I) = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{bmatrix}$

$|A - \lambda I| = [(3-\lambda)(-1-\lambda) - 0]$

$= -3 - 3\lambda + \lambda + \lambda^2 \Rightarrow \lambda^2 - 2\lambda - 3$

$= \lambda^2 - 3\lambda + \lambda - 3 = 0$

$\lambda(\lambda-3) + 1(\lambda-3) = 0$

$\lambda = 3$ & $\lambda = -1$

$\lambda_1 = 3, \quad \lambda_2 = -1$

$A^{25} = 3^{25}, (-1)^{25} \Rightarrow 3^{25}$ & -1 Ans

$A + 2I = 3 + 2\times1 = 5$
$-1 + 2\times1 = 1$ { Eigen Value of $I = 1$ }

Eigen Vectors

Find Eigen Value & Corresponding eigen Vectors of the matrix.

$$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

$$(A - \lambda I) = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix}$$

Characteristic Equation.

$$|A - \lambda I| = (-5-\lambda)(-2-\lambda) - 4$$
$$= +10 + 5\lambda + 2\lambda + \lambda^2 - 4 = \boxed{\lambda^2 + 7\lambda + 6 = 0}$$

$\lambda^2 + 6\lambda + \lambda + 6 = 0$

~~$\lambda(\lambda+5) + 2(\lambda+5) = 0$~~

~~$\lambda = -5, -2$~~ ~~$\lambda(\lambda+6)+$~~

$\lambda^2 + 7\lambda + 6 = 0$
$\lambda^2 + 6\lambda + \lambda + 6 = 0$
$\lambda(\lambda+6) + 1(\lambda+6) = 0$

$\lambda_1 = -1$
$\lambda_2 = -6$.

$AX = \lambda X$
$(A - \lambda I)X = 0$

Case I When $\lambda = -1$

$$\begin{bmatrix} -5-(-1) & 2 \\ 2 & -2+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-4x_1 + 2x_2 = 0$
$2x_1 - x_2 = 0$

~~Let x_1~~

$2x_1 = x_2$
$x_1 = \frac{x_2}{2}$

Let $x_1 = K$
$x_2 = 2K$.

$\lambda = -1$

$X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} K \\ 2K \end{bmatrix} \sim K \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Case 2 When $\lambda = -6$

$$\begin{bmatrix} -5-(-6) & 2 \\ 2 & -2-(-6) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 + 2x_2 = 0$ — (i)

$2x_1 + 4x_2 = 0$ — (ii)

from (i)

$x_1 = -2x_2$

~~Let $x_1 = k$~~ $x_1 = K_1$.

$x_2 = \frac{-K_1}{2}$.

Eigen vectors are.

$X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} K_1 \\ \frac{-K_1}{2} \end{bmatrix}$

$X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2K_1 \\ -K_1 \end{bmatrix}$ [$\because$ multiply by 2]

Q. Properties of eigen Vectors

(1) The eigen vector X of a matrix is not unique.

(2) If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct eigen value of $n \times n$ matrix, then corresponding eigen vectors $X_1, X_2, \ldots, X_n$ form a linearly independent set.

(iii) If two or more eigen Value are equal, It may or may not be possible to get linearly independent eigen Vectors, Corresponding to equal roots.

(iv) Two eigen Vectors x_1, x_2 are Called orthogonal Vectors if $\boxed{(x_1)^T x_2 = 0}$

Q Find the eigen Values, Eigen Vectors of the Corresponding

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & 2 & 0 \end{bmatrix}$$

$$(A - \lambda I) = \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix}$$

$$|A - \lambda I| = -1(\lambda+2)\left[-\lambda(1-\lambda) - 12\right] + 2\left[-2\lambda - 6\right] + 3\left[-4 + 1 - \lambda\right]$$

$$\cancel{3[}\ 3(\lambda+3) + 4(\lambda+3] + \lambda+2\ [\lambda^2 - \lambda - 12]$$

$$= ?.$$

$$\lambda^3 - P\lambda^2 + Q\lambda - |A| = 0$$

P = Sum of diagonal of A

Q = Sum of the minors of diagonal of A.

Q Find eigen Values & eigen Vectors of given eqⁿ.

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$2-\lambda[(2-\lambda)(2-\lambda)-1] + 1[-1(2-\lambda)+1] + 1[-1-(1)(2-\lambda)]$$

$$\lambda^3 - P\lambda^2 + Q\lambda - |A| = 0$$

P = Sum of diagonal element

Q = Sum of minors of diagonal elements.

$$P = 2+2+2 = 6$$

$$Q = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$Q = 4-1+4-1+4-1$$

$$Q = 9$$

$$|A| = 2\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + 1\begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix}$$

$$= 4$$

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0.$$

$(\lambda - 1)$ is factor of given Equation —

$\lambda - 1 \overline{) \lambda^3 - 6\lambda^2 + 9\lambda - 4} (\lambda^2 - 5\lambda + 4$

$\lambda^3 - \lambda^2$

$-5\lambda^2 + 9\lambda - 4$

$-5\lambda^2 + 5\lambda$

$4\lambda - 4$

$4\lambda - 4$

0

$(\lambda - 1)(\lambda^2 - 5\lambda + 4) = 0.$

$\lambda = 1, 1, 4$

<u>Short Cut trick to find λ</u>

λ^3
λ^2
λ
Constant

	λ^3	λ^2	λ	Constant
1	1	-6	9	-4
		1	-5	4
	1	-5	4	0

$(\lambda - 1)(\lambda^2 - 5\lambda + 4)$

$\lambda = 1, 1, 4.$

The eigen values of orthogonal matrix ~~will~~ for ~~be~~ λ will be equal.

$\boxed{X_1^T X_2 = 0}$

$\lambda = 4$

$$\begin{bmatrix} 2-4 & -1 & 1 \\ -1 & 2-4 & -1 \\ 1 & -1 & 2-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x_1 - x_2 + x_3 = 0$ — (i) Choose any
$-x_1 - 2x_2 - x_3 = 0$ — (ii) two equation
$x_1 - x_2 - 2x_3 = 0$ — (iii)

from eqⁿ (i) & (iii)

$-2x_1 - x_2 + x_3 = 0$
$x_1 - x_2 - 2x_3 = 0$

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ -1 & -2 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & -1 \\ 1 & -1 \end{vmatrix}}$$

$$\frac{x_1}{2+3} = \frac{x_2}{1-4} = \frac{x_3}{1+2}$$

$$\frac{x_1}{3} = \frac{x_2}{-3} = \frac{x_3}{3}$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1} = K$$

$x_1 = K$
$x_2 = -K$
$x_3 = K$

$$X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

When $\lambda = 1$

$\lambda^3 - P\lambda^2 + Q\lambda - |A| = 0$

$$\begin{bmatrix} 2-1 & -1 & 1 \\ -1 & 2-1 & -1 \\ 1 & -1 & 2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$x_1 - x_2 + x_3 = 0$

Put $x_1 = 0$

$-x_2 + x_3 = 0$

$-x_2 = -x_3$

$\frac{x_2}{1} = \frac{x_3}{1} = K$

$$X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

* for the Symmetric matrix, the eigen Vectors are orthogonal.

$X_1^T X_3 = 0$ let $X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$

$X_2^T X_3 = 0$

$X_1^T X_2 = 0$

$$X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$X_1^T X_3 = 0$

$$\begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$$

$l - m + n = 0$ —— (1)

$X_2^T X_3 = 0$

$$\begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$$

$0l + m + n = 0$ —— (2)

from eq-n (1) & (2)

$l - m + n = 0$

$0l + m + n = 0$

$$\frac{l}{\begin{vmatrix}-1 & 1\\ 1 & 1\end{vmatrix}} = \frac{m}{\begin{vmatrix}1 & 1\\ 0 & 1\end{vmatrix}} = \frac{n}{\begin{vmatrix}1 & -1\\ 0 & 1\end{vmatrix}}$$

$$\frac{l}{-1-1} = \frac{m}{0-1} = \frac{n}{1-0}$$

$$\frac{l}{-2} = \frac{m}{-1} = \frac{n}{1}$$

$$X_3 = \begin{bmatrix}-2\\ -1\\ 1\end{bmatrix}$$ Ans

Diagonalisation (Spectral Matrix)

Similarity Transformation
(Any type of Matrix)

$D = P^{-1}AP$

Steps

1. Find the eigen Values
2. Find the eigen Vectors
3. Form the Modal Matrix
   $P = [x_1\ x_2\ x_3]$
4. Find the inverse of P
   $P^{-1} = \frac{adj}{|P|}$
5. find $D = P^{-1}AP$.

Orthogonal Transformation
(Symmetric Matrix)

$D = N^T A N$

Steps

1. Find eigen Values
2. Find eigen Vectors
   $x_1\ x_2\ x_3$
3. Form normalised Modal Matrix,
   $N = \left| \frac{x_1}{|x_1|}, \frac{x_2}{|x_2|}, \frac{x_3}{|x_3|} \right|$
   $X_1 = \begin{bmatrix}a\\ b\\ c\end{bmatrix}$

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$|X_1| = \sqrt{a^2+b^2+c^2}$

(4) $N^T = ?$ (Transpose)

$D = N^T A N$

eg $X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ $|X_1| = \sqrt{1^2+2^2+2^2}$

$\sqrt{9} = 3$

Normalised Vector $= \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$

Diagonalise

$A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$

$|A - \lambda I| = 0$

$\begin{vmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = 0$

form $\lambda^2 - P\lambda + |A| = 0$

$(4-\lambda)(3-\lambda) - 2 = 0$

$\lambda^2 - 7\lambda + 10 = 0$

$\lambda^2 - 5\lambda - 2\lambda + 10 = 0$

$\lambda(\lambda-5) - 2(\lambda-5) = 0$

$\lambda = 5, 2$

$\lambda = 5$

$[A - \lambda I] X = 0$

$P = 4+7 = 7$

$|A| = 12 - 2 = 10$

$\begin{bmatrix} 4-5 & 1 \\ 2 & 3-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{matrix} 0 \\ 0 \end{matrix}$

$\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-x_1 + x_2 = 0$

$2x_1 - 2x_2 = 0$

Both eqn are same.

$-x_1 + x_2 = 0$

$-x_1 = -x_2$

$\frac{x_1}{1} = \frac{x_2}{1}$

$X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 2$

$$\begin{bmatrix} 4-2 & 1 \\ 2 & 3-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$2x_1 + x_2 = 0$$

$$2x_1 = -x_2$$

$$\frac{x_1}{1} = \frac{-x_2}{2} \Rightarrow \frac{x_1}{1} = \frac{x_2}{-2}$$

$$X_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

$$P^{-1} = \frac{\text{adj } P}{|P|}$$

$$\text{adj } P = \begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$|P| = -2 - 1 = -3$$

$$P^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{-1}{3} \end{bmatrix}$$

यहाँ multiply करने के बजाय Direct λ का Value Dale

$$D = P^{-1} A P$$

$$= -\frac{1}{3} \begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

Trick

$$= -\frac{1}{3} \begin{bmatrix} -15 & 0 \\ 0 & -6 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

If $\lambda = 5, 3, 2$

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Note: Power of Matrix Can be Calculated

$$\boxed{A^n = PD^n \cdot P^{-1}}$$

Q Diagonalise

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$$

$$|A| = 1\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ -4 & 3 \end{vmatrix} + 1\begin{vmatrix} 0 & 2 \\ -4 & 4 \end{vmatrix}$$

$|A| = 6-4 - 1(0+4) + 1(0+8)$

$|A| = 2 - 4 + 8$

$|A| = 10 - 4 = 6$

$(A - \lambda I) = 0$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ -4 & 4 & 3-\lambda \end{bmatrix}$$

$|A - \lambda I| = 0$

$$1-\lambda \begin{vmatrix} 2-\lambda & 1 \\ 4 & 3-\lambda \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ -4 & 3-\lambda \end{vmatrix} + 1\begin{vmatrix} 0 & 2-\lambda \\ -4 & 4 \end{vmatrix}$$

$1-\lambda\,[(2-\lambda)(3-\lambda) - 4] - 1[0+4] + 1[0 - (-4)(2-\lambda)]$

$(1-\lambda)[6 - 2\lambda - 3\lambda + \lambda^2 - 4] - 4 + 1[-\{-8 + 4\lambda\}]$

$(1-\lambda)(\lambda^2 - 5\lambda + 2) - 4 + 8 - 4\lambda$

$(1-\lambda)(\lambda^2 - 5\lambda - 2) + 4 - 4\lambda$

$\lambda^2 - 5\lambda - 2 - \lambda^3 + 5\lambda^2 + 2\lambda + 4 - 4\lambda = 0$

$-\lambda^3 + 6\lambda^2 - 7\lambda + 2 = 0$

$P = 1+2+3 = 6$

$Q = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix}$

$Q = 2-0 + 3+4 + 6-4$

$= 2 + 7 + 6 - 4$

$= 15 - 4 = 11$

$\lambda^3 - P\lambda^2 + Q\lambda - |A| = 0$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

find $\lambda = A$ & so on.

30-9-22

Leibnitz's theorem And its applications

Successive Differentiation

$y = f(x)$

$\frac{dy}{dx} = ?$

$y = \sin(\sin x)$ Prove that

$$\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} + y\cos^2 x = 0$$

$y = \sin(\sin x)$.

$\frac{dy}{dx} = -\cos(\sin x).-\cos x = \cos(\sin x)\cos x$.

$\frac{d^2y}{dx^2} = \cos(\sin x).-\sin x + \cos x.-\sin(\sin x).\cos x$

$\cos(\sin x) - \sin x - \cos^2 x \underline{\sin(\sin x)}$.

~~$\cos(\sin x) - \sin x$~~

$\frac{d^2y}{dx^2} = -\sin x\cos(\sin x) - y\cos^2 x$

$\frac{d^2y}{dx^2} + y.\cos^2 x = -\sin x\cos(\sin x)$

$= -\frac{\sin x}{\cos x}.\cos x\cos(\sin x)$

$\frac{d^2y}{dx^2} + y\cos^2 x = -\tan x \frac{dy}{dx}$.

$\frac{d^2y}{dx^2} + y\cos^2 x + \tan x \frac{dy}{dx} = 0$.

Proved

n^{th} Derivatives

$y = x^m$.

$y_1 = \frac{dy}{dx} = y' = mx^{m-1}$

$y_2 = m(m-1)x^{m-2}$

$y_3 = m(m-1)(m-2)x^{m-3}$

$y_n = m(m-1)(m-2)(m-3) \ldots [m-(n-1)]x^{m-n}$

$y_n = m(m-1)(m-2) \ldots [m-n+1)x^{m-n}$

(1) If $n = m$

$y_n = m(m-1)(m-2) \ldots [m-m+1]x^{m-m}$

$y_n = m(m-1)(m-2) \ldots x^0$

$\boxed{y_n = m!}$

Ex $y_{30} = y = x^{30}$

$= 30!$

(2) $y = e^{ax}$.

$y_1 = ae^{ax}$

$y_2 = a^2e^{ax}$

$\boxed{y_n = a^n e^{ax}}$

(3) $y = a^{mx}$

$y_n = a^x (\log_e a)^n$

$$\boxed{\frac{d}{dx}(a^x) = a^x \log_e a}$$

(4) $y = (ax+b)^{-1}$ or $\frac{1}{ax+b}$

$y_n = (-1)^n (ax+b)^{-n-1} a^n \cdot n!$

(5) $y = (ax+b)^{-m}$

$$\boxed{y_n = \frac{(-1)^n (m+n-1)!\, a^n}{(m-1)!\, (ax+b)^{m+n}}}$$

(6) $y = \log(ax+b)$

$$\boxed{y_n = \frac{(-1)^{n-1} (n-1)!\, a^n}{(ax+b)^n}}$$

Q. Find the n^{th} derivative of $\log(ax+x^2)$

$y = \log(ax+x^2)$

$y = \log x(a+x) \sim \log(ax+b)$

$a = 1$

$\cancel{b} = x$

$\cancel{y_n =}\ y = \log x + \log(a+x)$

$$y_n = \frac{(-1)^{n-1} \cdot (n-1)!\, 1^n}{x^n} + \frac{(-1)^{n-1} (n-1)!\, 1^n}{(a+x)^n}$$

Q

$y = \sin(ax+b)$

$y_1 = a\cos(ax+b)$

$= a\left[\sin(ax+b) + \frac{\pi}{2}\right]$

$y_2 = \left[a^2 \cos(ax+b) + \frac{\pi}{2}\right]$

$y. = a^2 \sin\left[ax+b+\frac{2\pi}{2}\right]$

$y_n = a^n \sin\left[ax+b+\frac{n\pi}{2}\right]$

Or

If $y = \cos(ax+b)$

$y_n = a^n \cos\left[ax+b+\frac{n\pi}{2}\right]$

$y = a^{mx}$

$y_n = m^n a^{mx} (\log a)^n.$

If $m = 1,$

$y = a^x$

$y_n = a^x (\log a)^n$

$y = \sin 2x \sin 3x$ find

$y_n = ?$

Form

$y = \sin(ax+b)$

$y_n = \left(a^n \sin(ax+b+\frac{n\pi}{2}\right)$

Use

$2\sin A \sin B = \cos A-B - \cos(A+B)$

$\sin A \sin B = \frac{1}{2} \cos(A-B) - \cos(A+B)$

$\cancel{y} = \frac{1}{2}\left[\cos x - \cos 5x\right]$

$\cancel{y = \cos(ax+b)}$

$\cancel{y_n = a^n \cos\left[ax+b+\frac{n\pi}{2}\right]}$

$\left.\begin{array}{l} a=1 \\ b=0 \end{array}\right] \cos x = \cos(ax+b)$

$\left.\begin{array}{l} a=5 \\ b=0 \end{array}\right\} \cos 5x = \cos(ax+b)$

$y_n = \frac{1}{2}\left[\left[1^n \cos\left(x+\frac{n\pi}{2}\right)\right] - 5^n \cos\left\{5x+\frac{n\pi}{2}\right\}\right]$

Q $y = \cos(ax+b)$

$y_n = a^n \cos\left\{ax+b+\frac{n\pi}{2}\right\}$

Q $y = \sin^3 x$

$y = 3\sin x - 4\sin^3 x = \sin 3x$

$4\sin^3 x = 3\sin x - \sin 3x$

$\sin^3 x = \frac{1}{4}\left[3\sin x - \sin 3x\right]$

$y_n = \frac{1}{4}\left[3 \cdot 1^n \sin\left(x+\frac{n\pi}{2}\right) - 3^n \sin\left(3x+\frac{n\pi}{2}\right)\right]$

Use of Partial Fraction

Working rule :

(1) $\frac{Px}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$

(2) $$\frac{px}{(x-a)^2(x-b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$$

(3) $$\frac{px}{(x-a)^3(x-b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3} + \frac{D}{x-b}$$

(4) $$\frac{px}{(x-a)(x-b)(px^2+qx+r)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{Cx+d}{px^2+qx+r}$$

Q $$y = \frac{1}{6x^2-5x+1}$$

Step 1 — factorise

$6x^2-3x-2x+1$

$3x(2x-1)-1(2x-1)=0$

$(2x-1)(3x-1)$

$$= \frac{1}{(2x-1)(3x-1)}$$

$$= \frac{A}{2x-1} + \frac{B}{3x-1}$$

$$\frac{1}{(2x-1)(3x-1)} = \frac{A}{(2x-1)} + \frac{B}{3x-1}$$

div $$\frac{(2x-1)(3x-1)}{(2x-1)(3x-1)} = \frac{A\cancel{(2x-1)}(3x-1)}{\cancel{(2x-1)}} + \frac{B(2x-1)\cancel{(3x-1)}}{\cancel{3x-1}}$$

$1 = A(3x-1) + B(2x-1)$

if $x = \frac{1}{3}$

$1 = A(0) + B\left(\frac{2}{3}-1\right)$

$1 = B\left(\frac{2-3}{3}\right)$

$1 = B\left(\frac{-1}{3}\right)$

$\boxed{B = -3}$

~~A~~

If $x = \frac{1}{2}$

$1 = A(3 \times \frac{1}{2} - 1) + 0$

$1 = A\left(\frac{3-2}{2}\right)$

$1 = A\left(\frac{1}{2}\right)$

$\boxed{A = 2}$

$\therefore \frac{2}{2x-1} + \frac{-3}{3x-1}$

$\frac{1}{6x^2 - 5x + 1} = \frac{2}{2x-1} + \frac{-3}{3x-1}$.

$2(2x-1)^{-1} - 3(3x-1)^{-1}$

Using formula.

$= 2 \cdot (-1)^n [2x-1]^{-n-1} \cdot 2^n \cdot n! - 3 \times (-1)^n (3x-1)^{-n-1} 3^n \cdot n!$.

$= (-1)^n 2^{n+1} (2x-1)^{-n-1} n! - 3^{n+1} (-1)^n (3x-1)^{-n-1} \cdot n!$.

$(ax+b)^{-1}$

$y_n = (-1)^n (ax+b)^{-n-1} \cdot a^n \cdot n!$

n^{th} derivative of $e^{ax}\sin(bx+c)$

$$y = e^{ax}\sin(bx+c)$$

$$y_n = e^{ax} r^n \sin(bx+c+n\alpha)$$

Where, $r^2 = a^2+b^2$ & $\tan\alpha = \frac{b}{a}$

Similarly,

$$y = e^{ax}\cos(bx+c)$$

$$y_n = e^{ax} r^n \cos(bx+c+n\alpha)$$

Where, $r^2 = a^2+b^2$ & $\tan\alpha = \frac{b}{a}$

If $y = e^{x}\sin^3 x$.

find y_n.

$$= \sin 3x = 3\sin x - 4\sin^3 x$$

$$4\sin^3 x = 3\sin x - \sin 3x$$

$$\sin^3 x = \frac{1}{4}[3\sin x - \sin 3x]$$

$$y = e^{x}\frac{1}{4}[3\sin x - \sin 3x]$$

$$= \frac{3}{4}e^{x}\sin x - \frac{1}{4}e^{x}\sin 3x$$

$$y_n = e^{ax}(a^2+b^2)^{\frac{n}{2}}\sin(bx+c+n\alpha)$$

$$= e^{x}(1+1)^{\frac{n}{2}}\sin(x+c+n\alpha)$$

$$= e^{x}2^{\frac{n}{2}}\sin\left(x+c+n\frac{\pi}{4}\right)$$

$r^2 = a^2+b^2$

$r = (1+9)^{\frac{1}{n}}$

$\sqrt{10}$

$\alpha = \tan^{-1}\frac{3}{1}$

$a=1$	$a=1$
$b=1$	$b=3$
$c=0$	$c=0$

$r^2 = a^2+b^2$

$r = (a^2+b^2)^{\frac{1}{2}}$

$r^n = (a^2+b^2)^{\frac{n}{2}}$

$\alpha = \tan^{-1}\frac{b}{a}$

$\alpha = \tan^{-1}\frac{1}{1}$

$\alpha = \frac{\pi}{4}$

$$y_n = \left[\left(\frac{3}{4}\cdot e^x\, 2^{\frac{n}{2}} \sin\left(x + c + \frac{n\pi}{4}\right) - \frac{1}{4} e^x\, 10^{\frac{n}{2}} \sin(3x + c + n\tan^{-1}3)\right)\right]$$

$$y_n = \frac{3}{4} e^x\, 2^{\frac{n}{2}} \sin\left(x + \frac{n\pi}{4}\right) - \frac{1}{4} e^x\, 10^{\frac{n}{2}} \sin(3x + n\tan^{-1}3)$$

Q. Find the n^{th} derivative of

$$\frac{x^2}{(x+2)(2x+3)}$$

$(x+2)(2x+3)$

$2x^2 + 4x + 3x + 6$

$2x^2 + 7x + 6$

$$\frac{x^2}{2x^2+7x+6}$$

$2x^2$

* If the degree of the numerator is equal or greater than the degree of denominator then we have to divide the num. by the den.

$$2x^2+7x+6 \overline{\big)\, x^2 \,}\ \frac{1}{2}$$

$$x^2 + \frac{7}{2}x + \cancel{\frac{6}{2}}\,3$$

$$- \quad -$$

$$-\frac{7}{2}x - 3$$

$$\frac{x^2}{(x+2)(2x+3)} = \frac{1}{2} - \frac{\left[\frac{7}{2}x + 3\right]}{2x^2+7x+6}$$

$$= \frac{1}{2} - \boxed{\frac{\frac{7}{2}x - 3}{(x+2)(2x+3)}}$$

Do it Partial fraction.

$$\frac{-\frac{7}{2}x - 3}{(x+2)(2x+3)} = \frac{A}{x+2} + \frac{B}{2x+3}$$

$$-\frac{7}{2}x - 3 = A(2x+3) + B(x+2)$$

~~take B = 0~~

$$\cancel{-\frac{7}{2}x - 3 = A(2x+3)}$$

$\frac{1}{2}$

$$\cancel{A = \frac{-\frac{7}{2}x - 3}{2x+3}}$$

$$\cancel{A = \frac{-7x-6}{2(2x+3)} = \frac{-7x-6}{4x+3}}$$

$$\cancel{A = \frac{-7x-2-6}{4x-2+3} = \frac{14-6}{-8+3} = \frac{8}{-5}}$$

$\cancel{x = -2}$.

If $x = -2$

$B = 0$.

$$-\frac{7}{2}(-2) - 3 = A(2\times(-2)+3) + B(-2+2)$$

$$\Rightarrow 7-3 = A(-4+3) + 0$$

$$4 = A(-1)$$

$$A = -4$$

If $x = -\frac{3}{2}$

$A = 0$.

$$-\frac{7}{2} \times \left(-\frac{3}{2}\right) - 3 = A(0) + B\left(-\frac{3}{2}+2\right)$$

$$\frac{21}{4} - 3 = 0 + B\left(\frac{-3+4}{2}\right)$$

$$\frac{21-12}{\cancel{4}_{2}} = B\left(\frac{1}{\cancel{2}}\right)$$

$$\frac{9}{2} = B$$

$$\frac{x^2}{(x+2)(2x+3)} = \frac{1}{2} - \frac{4}{x+2} + \frac{9}{2} \cdot \frac{1}{2x+3}$$

y_n

$(x+2)^{-1}$

$(ax+b)^{-1}$

$$\frac{x^2}{(x+2)(2x+3)} = \frac{1}{2} - \frac{4}{x+2} + \frac{9}{2} \cdot \frac{1}{2x+3}$$

$$= \frac{1}{2} - 4(x+2)^{-1} + \frac{9}{2}(2x+3)^{-1}$$

$$y_n = \frac{1}{2}(-4)(-1)^n (x+2)^{-n-1}(1)^n \cdot n! + \frac{9}{2}(-1)^n (2x+3)^{-n-1}(2)^n n!$$

7/10/22

Leibnitz's theorem.

VVI

If $u + v$ are functions of x, then

$$\frac{d^n}{dx^n}(uv) = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \ldots\ldots + {}^nC_n u v_n.$$

$$= u_n v + n u_{n-1} v_1 + \frac{n(n-1)}{2} u_{n-2} v_2 + \ldots\ldots + u \cdot v_n.$$

$${}^nC_r = \frac{n!}{r!\,(n-r)!}$$

${}^nC_n = 1$

${}^nC_0 = 1$

${}^nC_1 = n.$

$${}^nC_2 = \frac{n!}{2!\,(n-2)!} = \frac{n(n-1)}{2}$$

Differentiate n times

Q. $x^2 y_2 + x y_1 + y = 0$.

$D^n(x^2 y_2) + D^n(x y_1) + D^n(y) = 0$

$D^n(x^2 y_2)$

$U = y_2$, $V = x^2$

$\underline{U_n \cdot V}$

$\underline{U_{n-1} V_1}$

$\underline{U_{n-2} V_2}$

$U_{n-2} = y_n$ $V = x^2$

$U_{n-1} = y_{n+1}$ $V_1 = 2x$

$U_n = y_{n+2}$ $V_2 = 2$.

$D^n(x^2 y_2) = y_{n+2} x^2 + n y_{n+1} 2x + \cancel{y_{n}2} \frac{n(n-1)}{2} y_n 2$

$D^n(x y_1) =$

$U = y_1$ | $V = x$

$U_{n-1} = y_n$ | $V_1 = 1$

$U_n = y_{n+1}$ | $V_2 = 0$

$D^n(x y_1) = y_{n+1} x + n y_n$

$D^n(y) =$

$U = y$ | $V = 1$

$\cancel{U_{n+1}}$ | $V_1 = 0$

$U_n = y_n$

$D^n(y) = y_n +$

$x^2 y_{n+2} + 2xn y_{n+1} + n(n-1) y_n + y_{n+1} x + n y_n + y_n = 0$

$x^2 y_{n+2} + x y_{n+1}(2n+1) + [n^2 - n + n + 1] y_n = 0$

$$\boxed{x^2 y_{n+2} + (2n+1) x y_{n+1} + (n^2+1) y_n = 0}$$ n th Derivative

Q. If $y = a\cos(\log x) + b\sin(\log x)$

Show that

$x^2 y_{n+2} + (2n+1) x y_{n+1} + (n^2+1) y_n = 0$

$y = a\cos(\log x) + b\sin(\log x)$

$y_1 = -a\sin(\log x) \times \frac{1}{x} + b\cos(\log x) \times \frac{1}{x}$

$y_1 x = -a\sin(\log x) + b\cos(\log x)$

$y_1 + x y_2 = -a\cos(\log x)\frac{1}{x} - b\sin(\log x) \times \frac{1}{x}$

$x y_1 + x^2 y_2 = -[a\cos(\log x) + b\sin(\log x)]$

$= -y$

$x^2 y_2 + x y_1 + y = 0$

[Check I st Question Same thing. Solve as you Solved I st question]

Q. Differentiate n times.

If $y = x^n \log x$

Prove that

$$y_{n+1} = \frac{n!}{x}.$$

$\Rightarrow$

$$y = x^n \log x$$

$$y_1 = x^n \frac{1}{x} + \log x \, n x^{n-1}.$$

$$x y_1 = x^n + x \log x \, n x^{n-1}.$$

$$x y_1 = x^n + \log \, n x^n.$$

$$= x^n + n \boxed{x^n \log x}$$

$$y_1 x = x^n + ny$$

Apply liebnitz's theorem,

$$y_{n+1} x + n \cancel{y_{n}} = n! + \cancel{n y_n}$$

$$y_{n+1} x = n!$$

$$\boxed{y_{n+1} = \frac{n!}{x}}$$

Proved

$u = y_1$ | $v = x$

$u_{n-1} = y_n$ | $v_1 = 1$

$u_n = y_{n+1}$

$x^m = n$. if $m = n$

$x^n = n!$

If $y = \sin(m \sin^{-1} x)$ Prove that

$$(1-x^2) y_{n+2} - (2n+1) x y_{n+1} + (m^2 - n^2) y_n = 0$$

Partial Differentiation :

$y = f(x)$.

$z = f(x, y) \rightarrow \{x$ or y is independent Variable$\}$

$\downarrow$

Dependent Variable.

$\frac{\partial z}{\partial y} \rightarrow$ Differentiate with respect to (y) & Keeping (x) as Constant.

$\frac{\partial z}{\partial x} \rightarrow$ Differentiate with respect to (x) & Keeping (y) as Constant.

Ex. 1 $z = x^2 + y^2$ find $\frac{\partial z}{\partial x}$ or $\frac{\partial z}{\partial y}$?

Solution $z = x^2 + y^2$.

$$\frac{\partial z}{\partial x} = 2x + 0 = 2x.$$

$$\frac{\partial z}{\partial y} = 0 + 2y = 2y.$$

Ex. 2 $z = 3x^2y$ find $\frac{\partial z}{\partial x}$ or $\frac{\partial z}{\partial y}$?

Solution $z = 3x^2y$

$$\frac{\partial z}{\partial x} = 3y \cdot 2x = 6xy.$$

$$\frac{\partial z}{\partial y} = 3x^2 \cdot (1) = 3x^2.$$

Ex. 3 If $z = x^2 + y^2 - 3xy$ find $\frac{\partial z}{\partial x}$ or $\frac{\partial z}{\partial y}$;

Solution $z = x^2 + y^2 - 3xy$.

$$\frac{\partial z}{\partial x} = 2x + 0 - 3y = 2x - 3y$$

$$\frac{\partial z}{\partial y} = 0 + 2y - 3x = 2y - 3x.$$

Ex. 4. If $z = \log(x^2+y^2)$ find $\frac{\partial z}{\partial x}$ or $\frac{\partial z}{\partial y}$?

Solution $\frac{\partial z}{\partial x} = \frac{1}{x^2+y^2} \times 2x = \frac{2x}{x^2+y^2}$

$$\frac{\partial z}{\partial y} = \frac{1}{x^2+y^2} \times 2y = \frac{2y}{x^2+y^2}$$

Ex. 5 $F = \sin(5x^2y + 7xy^2)$ find $\frac{\partial f}{\partial x}$ or $\frac{\partial F}{\partial y}$?

Solution

$$F = \sin(5x^2y + 7xy^2)$$

$$\frac{\partial F}{\partial x} = \cos(5x^2y + 7xy^2)$$

$$= \cos\{5y \cdot 2x + 7y^2 \cdot (1)\}$$

$$\frac{\partial F}{\partial x} = \cos[5x^2y + 7xy^2](10xy + 7y^2)$$

$$\frac{\partial F}{\partial y} = \cos(5x^2y + 7xy^2)$$

$$= \cos(5x^2 + 2y \cdot 7x)$$

$$\frac{\partial f}{\partial y} = \cos[5x^2y + 7xy^2](5x^2 + 14xy)$$

Ex. 6 If $f = x^4 \sin(xy^3)$ find $\frac{\partial f}{\partial x}$ or $\frac{\partial f}{\partial y}$?

Solution $f = x^4 \sin(xy^3)$

$$\frac{\partial f}{\partial x} = x^4 \cos(xy^3) \cdot y^3 + \sin(xy^3) \cdot 4x^3.$$

$$\frac{\partial f}{\partial y} = x^4 \cos(xy^3) \cdot 3y^2 \cdot x \cdot \cancel{+ \sin(xy^3) \cdot x^4}$$

$$\frac{\partial f}{\partial y} = 3x^5 y^2 \cos(xy^3)$$

$$\left\{\because d(uv) = u \cdot d(v) + v \cdot d(u)\right\}$$

Q.1. If $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$.
find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$?

Solution

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\left(\frac{1}{y}\right) + \frac{1}{1+\left(\frac{y}{x}\right)^2} \times y \times \frac{-1}{x^2}.$$

Differentiation $\tan^{-1} x$.

$$\frac{\partial}{\partial x}(\tan^{-1} x) = \frac{1}{1+x^2}$$

Differentiation $\sin^{-1} x$.

$$\frac{\partial}{\partial x}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{1}{\sqrt{\frac{y^2-x^2}{y^2}}} \times \frac{1}{y} + \frac{1}{\frac{x^2+y^2}{x^2}} \times y \times \frac{-1}{x^2}$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2}.$$

$$x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} \quad \text{— eq}^n \ 1$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\, x \times -\frac{1}{y^2} + \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \frac{1}{x}.$$

$$= \frac{1}{\sqrt{\frac{y^2-x^2}{y^2}}} \times x \times \frac{-1}{y^2} + \frac{x^2}{x^2+y^2} \times \frac{1}{x}$$

$$\frac{\partial u}{\partial y} = \frac{-x}{y}\frac{1}{\sqrt{y^2-x^2}} + \frac{3}{x^2+y^2}$$

$$y\frac{\partial u}{\partial y} = \frac{-x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2} \quad \text{— eq}^n \ 2$$

∴ Now adding eq^n 1 & Eq^n 2.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \cancel{\frac{x}{\sqrt{y^2-x^2}}} + \cancel{\frac{xy}{x^2+y^2}} - \cancel{\frac{x}{\sqrt{y^2-x^2}}} + \cancel{\frac{xy}{x^2+y^2}}$$

$$= 0.$$

$$\therefore \boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0}$$

13-10-22

Homogeneous function.

A function $f(x,y)$ is said to be Homogeneous function in which the power of each term is same.

A function $f(x)$ is a homogeneous function of order "n" if degree of each of its term in x, y is equal to n.

$$a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \dots a_{n-1}xy^{n-1} + a_ny^n.$$

is homogeneous function of order n.

$$x^n\left[a_0 + a_1\left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + \dots\dots + a_n\left(\frac{y}{x}\right)^n\right] = x^n \phi\left(\frac{y}{x}\right)$$

for ex.

$$\frac{\sqrt{x}+\sqrt{y}}{x^2+y^2} = \frac{\sqrt{x}\left[1+\sqrt{\frac{y}{x}}\right]}{x^2\left[1+\left(\frac{y}{x}\right)^2\right]}$$

$$= \frac{x^{\frac{1}{2}}\left[1+\sqrt{\frac{y}{x}}\right]}{x^2\left[1+\left(\frac{y}{x}\right)^2\right]}$$

$$= x^{-\frac{3}{2}}\left[\frac{1+\sqrt{\frac{y}{x}}}{1+\left(\frac{y}{x}\right)^2}\right]$$

Order $= n = -\frac{3}{2}$

***

Euler's theorem on homogeneous function:

If z is homogeneous function is x, y order n.

then $$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz.$$

Verify Euler's theorem for the function.

$$u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}.$$

Solution

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \times \frac{1}{y} + \frac{1}{1+\left(\frac{y}{x}\right)^2} \times -\frac{1}{x^2} \cdot y$$

$$= \frac{1}{\sqrt{\frac{y^2-x^2}{y^2}}} \cdot \frac{1}{y} + \frac{1}{\frac{x^2+y^2}{x^2}} \times -\frac{1}{x^2} \times y$$

$$= \frac{1}{\sqrt{y^2-x^2}} + \frac{1 \times -y}{x^2+y^2}$$

$$= \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}$$

$$x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} \quad \text{— (i)}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \times x\left(-\frac{1}{y^2}\right) + \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \frac{1}{x}$$

$$= \frac{1}{\sqrt{\frac{y^2-x^2}{y^2}}} \times -\frac{x}{y^2} + \frac{1}{\frac{x^2+y^2}{x^2}} \times \frac{1}{x}.$$

$$= \frac{-x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}$$

$$y\frac{\partial u}{\partial y} = \frac{-x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2} \quad \text{— (ii)}$$

Add (i) & (ii)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \cancel{\frac{+x}{\sqrt{y^2-x^2}}} - \cancel{\frac{xy}{x^2+y^2}} + \left(\cancel{\frac{-x}{\sqrt{y^2-x^2}}} + \cancel{\frac{xy}{x^2+y^2}}\right)$$

$$= 0$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

$$u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}.$$

$$= x^{0}\sin^{-1}\frac{x}{y} + x^{0}\tan^{-1}\frac{y}{x}$$

$$n = 0$$

By Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

Deduction from Euler's theorem: ***** VVI.

If z is a homogeneous function of x, y of degree and $z = f(u)$; then.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{nf(u)}{f'(u)}$$

If $u = \log\left(\frac{x^2+y^2}{x+y}\right)$ Prove that

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1}$$

Solution $u = \log\left(\frac{x^2+y^2}{x+y}\right)$

$$\left\{ \because \begin{array}{l} u = \log x \\ x = e^{u} \end{array} \right\}$$

$$= \frac{x^2+y^2}{x+y} = e^{u}$$

$$z = e^{u} = \frac{x^2+y^2}{x+y}$$

$$= \frac{x^2\left[1+\left(\frac{y^2}{x^2}\right)\right]}{x\left[1+\frac{y}{x}\right]}$$

$$= x\left[\frac{1+\left(\frac{y}{x}\right)^2}{1+\frac{y}{x}}\right]$$

yes it is Homogeneous

$n = 1.$

By Euler's deduction formula,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{fu}{f'(u)} = 1 \cdot \frac{\cancel{e^u}}{\cancel{e^u}} = 1$$

2nd deduction formula.

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)\left[g'(u) - 1\right]$$

Where,

$$g(u) = \frac{nf(u)}{f'(u)}$$

Ist deduction formula:

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{nfu}{f'(u)}$$

If $u = \tan^{-1}\left(\frac{x^3+y^3}{\sqrt{x}+\sqrt{y}}\right)$ find the Value of.

(1) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

(2) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$

Q $u = \tan^{-1}\left(\frac{x^3+y^3}{\sqrt{x}+\sqrt{y}}\right)$

$$\frac{x^3+y^3}{\sqrt{x}+\sqrt{y}} = \tan u.$$

$$\frac{x^3\left(1 + \frac{y^3}{x^3}\right)}{\sqrt{x}\left(1+\sqrt{\frac{y}{x}}\right)} = \tan u$$

$$x^{3-\frac{1}{2}}\left[\frac{1+\left(\frac{y}{x}\right)^3}{1+\left(\frac{y}{x}\right)^{\frac{1}{2}}}\right]$$

$$x^{\frac{5}{2}}\left[\frac{1+\left(\frac{y}{x}\right)^2}{1+\left(\frac{y}{x}\right)^{\frac{1}{2}}}\right] \qquad n=\frac{5}{2}.$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{nf(u)}{f'(u)}$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{5}{2}\frac{\tan u}{\sec^2 u}$$

$$=\frac{5}{2}\sin u\cos u.$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{5}{2}\times\frac{1}{2}\times 2\sin u\cos u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{5}{4}\sin 2u$$

Now $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+x^2\frac{\partial^2 u}{\partial y^2}=g(u)\left[g'(u)-1\right]$$

$$=\frac{5}{4}\sin 2u\left[\frac{5}{2}\cos 2u\times 2-1\right]$$

$$=\frac{25}{8}\sin 2u\cos u-\frac{5}{4}\sin 2u$$

Ans

Total derivative

(i) Differentiation

(ii) Partial differentiation

(iii) total differentiation

$y=f(x)$.

$\frac{\partial y}{\partial x}$. $\frac{dz}{dx}$ $\frac{dz}{dy}$

If $z = f(x, y)$.
$x = \phi(t)$
$y = \phi(t)$.

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x} \times \frac{dx}{dt} + \frac{\partial z}{\partial y} \times \frac{dy}{dt}}$$

$v = x^2$

$\frac{dv}{dx} = 2x$

$\cancel{\frac{\partial v}{\partial y} = 2x}$.

Q If $z = x^2 y$ Where $x = t^2$, $y = t^3$
find $\frac{dz}{dt}$.

Solution $z = x^2 y$.

$\frac{\partial z}{\partial x} = 2xy$. ; $\frac{\partial z}{\partial y} = x^2$

then, $x = t^2$; $y = t^3$

$\frac{dx}{dt} = 2t$; $\frac{dy}{dt} = 3t^2$.

Now $\frac{dz}{dt} = \frac{\partial z}{\partial x} \times \frac{dx}{dt} + \frac{\partial z}{\partial y} \times \frac{dy}{dt}$

$= 2xy \cdot 2t + x^2 \cdot 3t^2$

$= 4xyt + 3x^2 t^2$

Now put the value of x & y.

$\frac{dz}{dt} = 4(t^2)(t^3) t + 3(t^2)^2 t^2$

$4t^6 + 3t^6 = 7t^6$. Ans

Q. If $U = x^2 + y^2$.
Where, $x = at^2$ $y = 2at$. find $\frac{du}{dt}$.

$U = x^2 + y^2$. ; $\frac{\partial u}{\partial y} = 0 + 2y$.
$\frac{\partial u}{\partial x} = 2x + 0$

then, $x = at^2$; $y = 2at$
$\frac{dx}{dt} = a \cdot 2t = 2at$; $\frac{dy}{dt} = 2a$.

Now,
$$\frac{du}{dt} = \frac{\partial u}{\partial x} \times \frac{dx}{dt} + \frac{\partial u}{\partial y} \times \frac{dy}{dt}$$
$$\frac{du}{dt} = 2x \times 2at + 2y \times 2a$$
$$= 4atx + 4ay.$$
Put the values of ~~x~~ & y
$$\frac{du}{dt} = 4at(at^2) + 4a(2at)$$
$$\boxed{\frac{du}{dt} = 4a^2t^3 + 8a^2t}$$ Ans.

Q. If $U = x^2 + y^2 + z^2$. Where, $x = e^t$
$y = e^t \sin t$
$z = e^t \cos t$
Prove that.
$$\frac{du}{dt} = 4e^{2t}.$$

formula
$$\frac{du}{dt} = \frac{\partial u}{\partial x} \times \frac{dx}{dt} + \frac{\partial u}{\partial y} \times \frac{dy}{dt} + \frac{\partial u}{\partial z} \times \frac{dz}{dt}.$$

Solution → $U = x^2 + y^2 + z^2$.
$\frac{\partial u}{\partial x} = 2x$ | $\frac{\partial u}{\partial y} = 0 + 2y$ | $\frac{\partial u}{\partial z} = 2z$

$x = e^t$ $\qquad$ $y = e^t \cdot \sin t$ $\qquad$ $z = e^t \cos t$

$\frac{dx}{dt} = e^t$ $\qquad$ $\frac{dy}{dt} = \cancel{e^t \sin t \cdot \cos t}$ $\qquad$ $\frac{dz}{dt} = e^t \cdot (-\sin t) + \cos t \, e^t$

$e^t \cdot \cos t + \sin t \, e^t$

$= e^t \cos t + e^t \sin t$ $\qquad$ $= e^t \sin t + e^t \cos t$

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \times \frac{dx}{dt} + \frac{\partial u}{\partial y} \times \frac{dy}{dt} + \frac{\partial u}{\partial z} \times \frac{dz}{dt}$$

$$= 2x\, e^t + 2y\left(e^t \cos t + e^t \sin t\right) + 2z \left(-e^t \sin t + e^t \cos t\right)$$

$$= 2xe^t + 2ye^t\left[\cos t + \sin t\right] - 2ze^t\left[\sin t - \cos t\right]$$

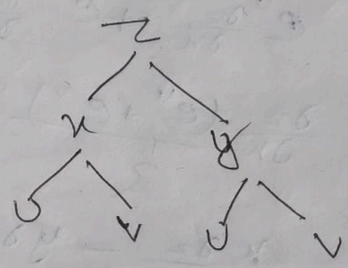
20-10-22

Change in the independent Variables $x+y$ by other two Variables $u+v$.

$$z = f(x,y).$$

Where. $x = \phi(u,v)$. & $y = \psi(u,v)$

$\cancel{\frac{\partial z}{\partial u}}$



$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial v}.$$

If $z = f(x,y)$, $x = e^u + e^{-v}$, $y = e^{-u} - e^v$

Show that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$.

$=$ $x = e^u + e^{-v}$ | $\frac{\partial x}{\partial v} = -e^{-v}$

$\frac{\partial x}{\partial u} = e^u$

$y = e^{-u} - e^v$ | $\frac{\partial y}{\partial v} = -e^v$

$\frac{\partial y}{\partial u} = -e^{-u}$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \times (e^u) + \frac{\partial z}{\partial y} \times (-e^{-u})$$

$$= e^u \frac{\partial z}{\partial x} - e^{-u} \frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\left(-e^{-v}\right) + \frac{\partial z}{\partial y}\left(-e^{v}\right)$$

$$= -e^{-v}\frac{\partial z}{\partial x} - e^{v}\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = e^{u}\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y} - \left[-e^{-v}\frac{\partial z}{\partial x} - e^{-v}\frac{\partial z}{\partial y}\right]$$

$$= e^{u}\frac{\partial z}{\partial x} + e^{-v}\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y} + e^{-v}\frac{\partial z}{\partial y}$$

$$= \frac{\partial z}{\partial x}\underbrace{\left(e^{u}+e^{-v}\right)}_{x} - \frac{\partial z}{\partial y}\underbrace{\left(e^{-u}-e^{+v}\right)}_{y}$$

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$ Proved.

If $u = u(y-z,\ z-x,\ x-y)$

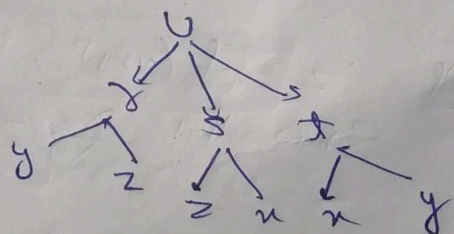
Prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Let

$r = y-z$

$s = z-x$

$t = x-y$



$u = u(r, s, t)$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\times\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\times\frac{\partial t}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\times\frac{\partial r}{\partial y} + \frac{\partial u}{\partial t}\times\frac{\partial t}{\partial y}.$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\times\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\times\frac{\partial s}{\partial z}.$$

$r = y-z$

$\frac{\partial r}{\partial y} = 1$

$\frac{\partial r}{\partial z} = -1$

$s = z-x$

$\frac{\partial s}{\partial z} = 1$

$\frac{\partial s}{\partial x} = -1$

$t = x-y$

$\frac{\partial t}{\partial x} = 1$

$\frac{\partial t}{\partial y} = -1$

$$\frac{\partial u}{\partial x} = \cancel{\frac{\partial u}{\partial s}(-1)} + \cancel{\frac{\partial u}{\partial t}(1)} \quad \text{— (i)}$$

$$\frac{\partial u}{\partial y} = \cancel{\frac{\partial u}{\partial r}(1)} + \cancel{\frac{\partial u}{\partial t}(-1)} \quad \text{— (ii)}$$

$$\frac{\partial u}{\partial z} = \cancel{\frac{\partial u}{\partial r}(-1)} + \cancel{\frac{\partial u}{\partial s}(1)} \quad \text{— (iii)}$$

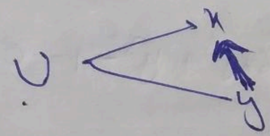
Add (i) + (ii) + (iii).

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0 \text{. Proved}$$

Implicit function

We cannot subtract $x - y$ because $x^3 = 1 - y^3 - 3xy$

Q If $u = x \log(xy)$, $x^3 + y^3 + 3xy = 1$. find $\frac{du}{dx}$.



$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$ for implicit function

$$\frac{\partial u}{\partial x} = x \cdot \frac{1}{xy} y + \log(xy) \times 1$$

$$= 1 + \log xy$$

$$\frac{\partial u}{\partial y} = x \cdot \frac{1}{xy} = \frac{x^2}{xy}$$

$$x^3 + y^3 + 3xy = 1.$$

$$3x^2 + 3y^2 \frac{dy}{dx} + 3x \frac{dy}{dx} + 3y = 0.$$

$$\frac{dy}{dx} = \frac{-x^2 + y}{x + y^2}$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$

$$= 1 + \log xy + \left(\frac{x^2}{xy}\right)\left(\frac{-x^2+y}{x+y^2}\right)$$

Let $z = f(x, y)$

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y}.$$

$\frac{\partial f}{\partial x} = p$ $\frac{\partial f}{\partial y} = q$ $\frac{\partial^2 f}{\partial x^2} = r$ $\frac{\partial^2 f}{\partial x \partial y} = s$

$\frac{\partial^2 f}{\partial y^2} = t$.

$$\boxed{\frac{d^2y}{dx^2} = \frac{-q^2 r - 2pqs + p^2 t}{q^3}}$$

Q. If $x^3 + 3x^2y + 6xy^2 + y^3 - 1 = 0$, find $\frac{dy}{dx}$.

$f = x^3 + 3x^2y + 6xy^2 + y^3 - 1$

$$\frac{\partial f}{\partial x} = 3x^2 + 6xy + 6y^2$$

$$\frac{\partial f}{\partial y} = 3x^2 + 12xy + 3y^2$$

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y}$$

$$= -\frac{3x^2 + 6xy + 6y^2}{3x^2 + 12xy + 3y^2}$$

$$= -\left(\frac{x^2 + 2xy + 2y^2}{x^2 + 4xy + y^2}\right)$$

Some Typical Cases.

Q. I If $x = u^2 - v^2$ and $y = uv$
find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$.

$$x = u^2 - v^2 \quad \text{— (1)}$$

diff w.r.t x

$$1 = 2u\frac{\partial u}{\partial x} - 2v\frac{\partial v}{\partial x}.$$

$$y = uv \quad \text{— (2)}$$

$$\cancel{\frac{\partial y}{\partial x}} = 0 = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}.$$

from eqn (1)

$$1 = 2u\frac{\partial u}{\partial x} - 2v\frac{\partial v}{\partial x} \quad \text{— (3)} \quad \times u.$$

$$0 = v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x} \quad \text{— (4)} \quad \times 2v.$$

$$u = 2u^2\frac{\partial u}{\partial x} - 2uv\frac{\partial v}{\partial x}$$

$$+ \quad 0 = 2v^2\frac{\partial u}{\partial x} + 2uv\frac{\partial v}{\partial x}.$$

$$u + 0 = 2u^2\frac{\partial u}{\partial x} + 2v^2\frac{\partial u}{\partial x}$$

$$u = (2u^2 + 2v^2)\frac{\partial u}{\partial x}.$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{u}{2u^2 + 2v^2}}$$

Now

$$1 = 2u\frac{\partial u}{\partial x} - 2v\frac{\partial v}{\partial x} \quad -③ \quad \times v$$

$$0 = v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x} \quad -④ \quad \times 2u.$$

$$v = \cancel{2uv\frac{\partial u}{\partial x}} - 2v^2\frac{\partial v}{\partial x}$$

$$- \quad 0 = \cancel{2uv\frac{\partial u}{\partial x}} + 2u^2\frac{\partial v}{\partial x}$$

$$v - 0 = -2v^2\frac{\partial v}{\partial x} - 2u^2\frac{\partial v}{\partial x}.$$

$$v = \cancel{-2v^2\partial x}\left(-2v^2 - 2u^2\right)\frac{\partial v}{\partial x}.$$

$$\boxed{\frac{\partial v}{\partial x} = \frac{v}{-2v^2 - 2u^2}}$$

$$\frac{\partial v}{\partial x} = -\frac{v}{2v^2 + 2u^2}$$

Again

$$x = u^2 - v^2$$

diff with respect to y

$$0 = 2u\frac{\partial u}{\partial y} - 2v\frac{\partial v}{\partial y}. \quad -⑤$$

$$y = u \cdot v.$$

diff w.r.t to y

$$1 = u.\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}. \quad -⑥$$

$\cancel{0 = 2v\partial}$ $$1 = u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y} \quad -⑥ \quad \cancel{\times 2v} \quad \times 2v$$

$$0 = 2u\frac{\partial u}{\partial y} - 2v\frac{\partial v}{\partial y} \quad -⑦ \quad \cancel{\times v} \quad \times u$$

$$\cancel{0 = 2uv\frac{\partial u}{\partial y} - 2v^2\frac{\partial v}{\partial y}}$$

$$\cancel{2v = 2uv\frac{\partial v}{\partial y} + 2v^2\frac{\partial u}{\partial y}}.$$

$$2v = \cancel{2uv\frac{\partial v}{\partial y}} + 2v^2\frac{\partial u}{\partial y}$$

$$0 = \cancel{-2uv\frac{\partial v}{\partial y}} + 2u^2\frac{\partial u}{\partial y}.$$

$$2v = (2v^2+2u^2)\frac{\partial u}{\partial y}$$

$$v = \frac{\partial u}{\partial y} = \frac{2v}{2v^2+2u^2} = \frac{v}{u^2+v^2}.$$

Again from equation 5 & 6.

$$0 = 2u\frac{\partial u}{\partial y} - 2v\frac{\partial v}{\partial y} \quad \times v$$

$$1 = \cancel{u\frac{\partial v}{\partial y}} \; v\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial y} \quad \times 2u.$$

$$0 = 2uv\frac{\partial u}{\partial y} - 2v^2\frac{\partial v}{\partial y}$$

$$2u = 2uv\frac{\partial u}{\partial y} + 2u^2\frac{\partial v}{\partial y}.$$

$$-2u = -2v^2\frac{\partial v}{\partial y} - 2u^2\frac{\partial v}{\partial y}$$

$$2u = 2v^2\frac{\partial v}{\partial y} + 2u^2\frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial y} = \frac{2u}{2v^2+2u^2} = \frac{u}{u^2+v^2}$$

Equation to tangent plane.

$$(x-x_1)\frac{\partial F}{\partial x} + (y-y_1)\frac{\partial F}{\partial y} + (z-z_1)\frac{\partial F}{\partial z} = 0.$$

Equation of normal to the plane.

$$\frac{x-x_1}{\frac{\partial F}{\partial x}} = \frac{y-y_1}{\frac{\partial F}{\partial y}} = \frac{z-z_1}{\frac{\partial F}{\partial z}}$$

Find the equation of the tangent plane and normal line to the surface.

$$x^2+2y^2+3z^2 = 12 \text{ at } (1,2,-1)$$

$$F = x^2+2y^2+3z^2-12 = 0.$$

$$\frac{\partial F}{\partial x} = 2x \quad \text{at } (1,2,-1) \quad \frac{\partial F}{\partial z} = 6z$$

$$\frac{\partial F}{\partial y} = 4y$$

$$(x_1,y_1,z_1) = (1,2,-1)$$

$$\frac{\partial F}{\partial x} \text{ at } (1,2,-1) = 2$$

$$\frac{\partial F}{\partial y} \text{ at } (1,2,-1) = 8$$

$$\frac{\partial F}{\partial z} \text{ at } (1,2,-1) = -6$$

Equation to tangent :

$$(x-1)\times 2 + (y-2)8 + (z+1)\times -6 = 0$$

$$2x-2+8y-16-6z-6 = 0$$

$$2x+8y-6z-24 = 0$$

$$x+4y-3z = 12$$

Equation to normal :

$$\frac{x-1}{2} = \frac{y-2}{8} = \frac{z+1}{-6}$$

$$\frac{x-1}{1} = \frac{y-2}{4} = \frac{z+1}{-3}$$